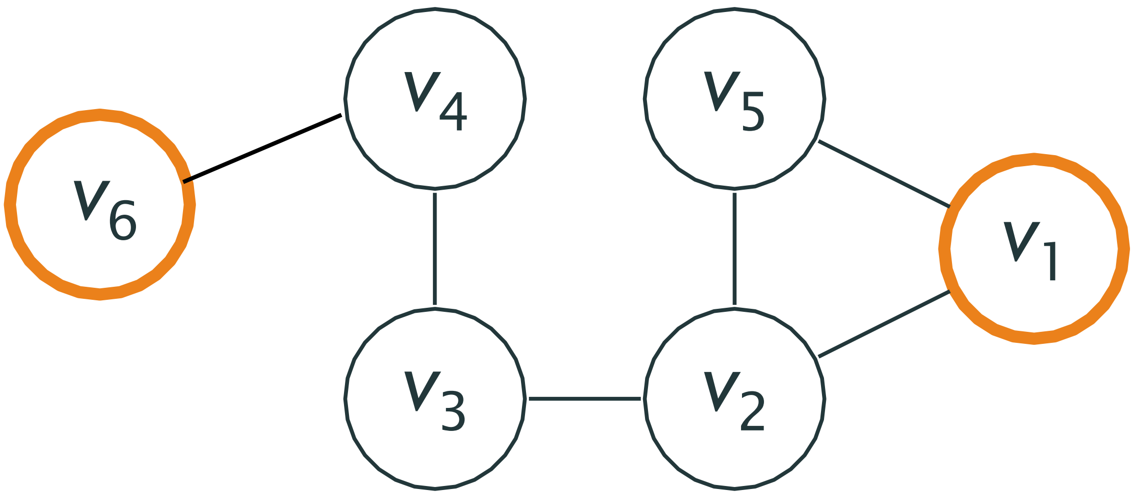
Graph Applications: Minimum-cost Spanning Trees

Lecture 28

by Marina Barsky

Recall: Connectivity in undirected graphs

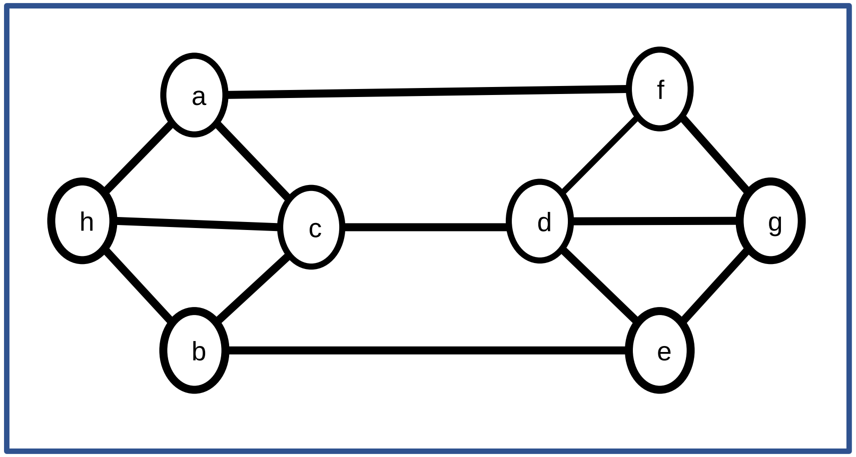
- Two vertices are connected, if there is a path between them



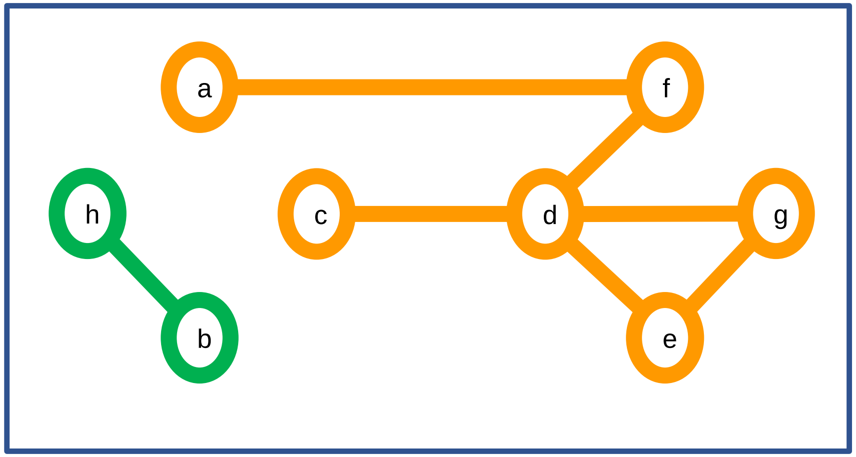
v_1 and v_6 are connected.

Recall: Connected graph

- A graph is connected, if any two of its nodes are connected. In other words, there is a path between any pair of nodes



This graph is connected.



This graph is not connected.

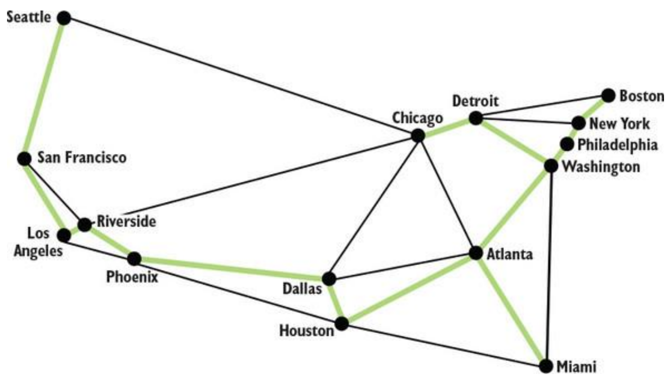
How to find if the graph is connected

1. How to find out whether an undirected Graph is connected?

- Hint: traversals
- What is the running time of these algorithms?

Minimum spanning trees: Motivation

- Connect all the computers in a new office building using the least amount of cable

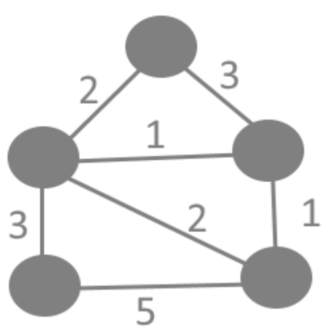


- Road repair: repair only min-cost roads such that all the cities are still connected
- Airline: downsize operations but preserve connectivity

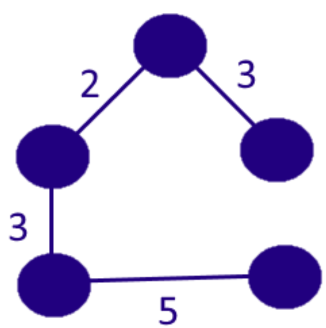
Definitions

- A Spanning Tree of a graph *G*, is a subgraph of *G* which is a tree and contains all vertices of *G*

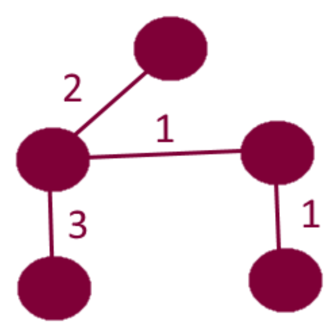
- A Minimum Spanning Tree (MST) of a **weighted** graph *G* is a spanning tree with the smallest total weight



Graph



Spanning Tree
Cost = 13



Minimum Spanning
Tree, Cost = 7

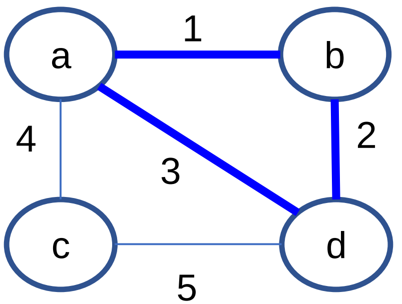
Problem: compute MST of Graph G

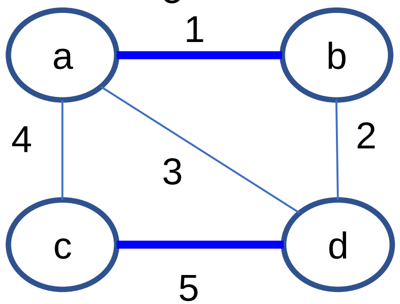
Input: undirected graph G=(V, E) and the weight w_e for each edge

Output: minimum-cost tree $T \in$ E that spans all the vertices V

Simplifying assumptions:

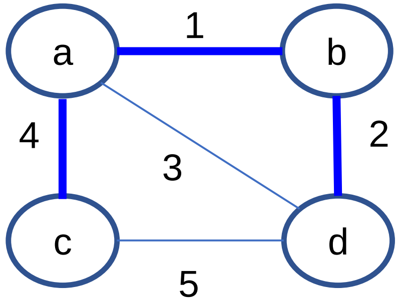
- *G* is undirected and simple (that is, it has no self-loops and no parallel edges)
- Input graph G is connected



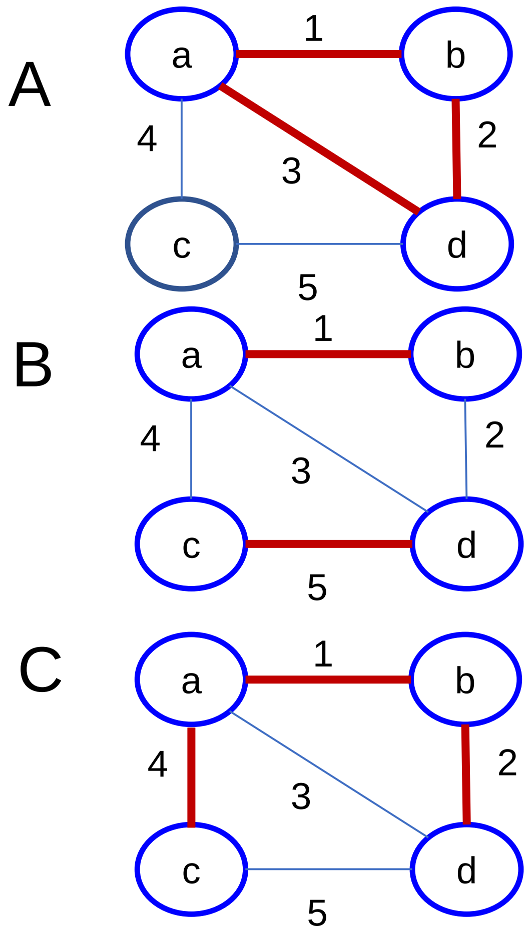


Tree means a subgraph that:

- ❑ has no cycles
- ❑ has exactly n-1 edges
- ❑ is connected



Which of the following subgraphs (in red) are Spanning trees



- A
- B
- C
- More than one of the above
- None of the above

Problem: compute MST of Graph G

Input: undirected graph G=(V, E) and the weight w_e for each edge

Output: minimum-cost tree $T \in$ E that spans all the vertices V

Simplifying assumptions:

- *G* is undirected and simple (that is, it has no self-loops and no parallel edges)
- Input graph G is connected

Tree means a subgraph that:

- ❑ has no cycles
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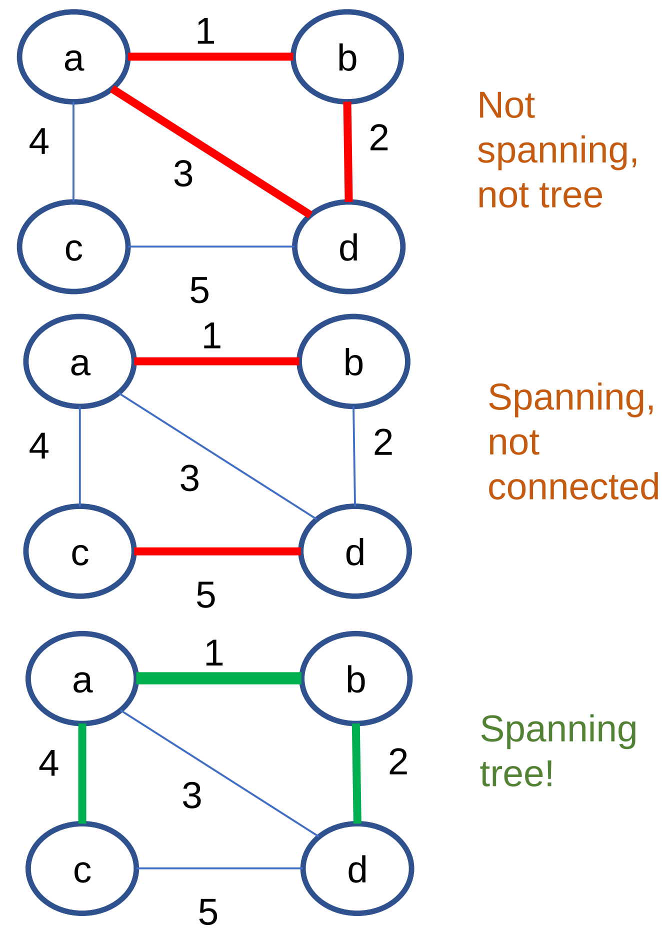
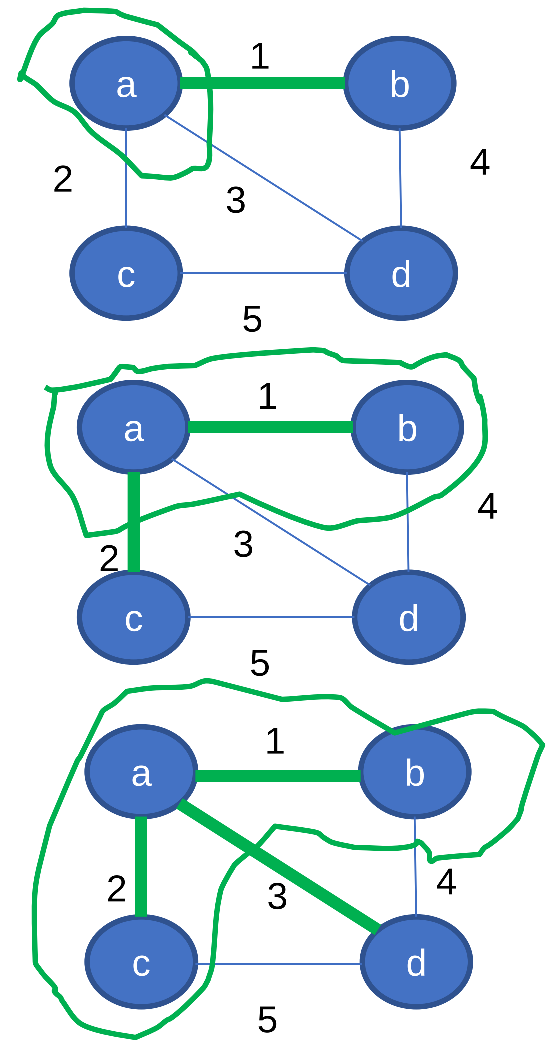


MST Algorithm by Prim

Grows a tree starting from a single (arbitrarily selected) vertex.

- Start from an arbitrary vertex
- Span another vertex by choosing **the edge with the min cost** **(greedy move)**
- Now have a tree of 2 vertices
- Check all edges out of this tree and choose the one with min-cost …



Algorithm Prim_MST (graph G(V,E))

Wy do we need edges only?

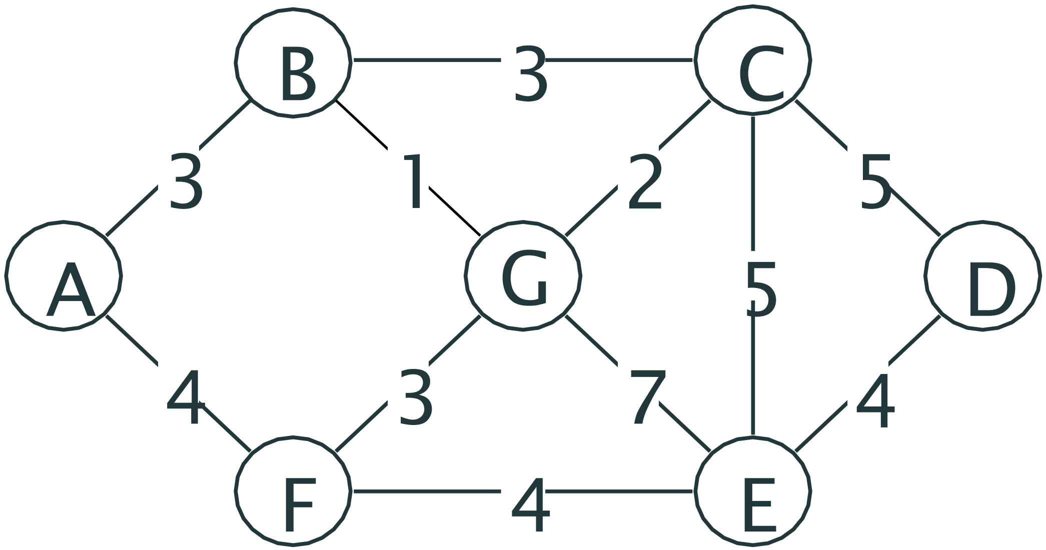
initialize tree T: = ∅ # set of tree ***edges***

X: = {vertex s} # s ∈ V, chosen arbitrarily

X contains vertices spanned by the tree-so-far

```
while |X|!=|V|:
    let e=(u,v) be the cheapest edge of G with u ∈ X and v ∉ X
    add e to T
    add v to X
    # that increases the number of spanned vertices
```

Which of the following sets of tree edges can be produced by the Prim algorithm?



A. (AB), (BG), (GC), (BC), (GF), (CD), (FE)

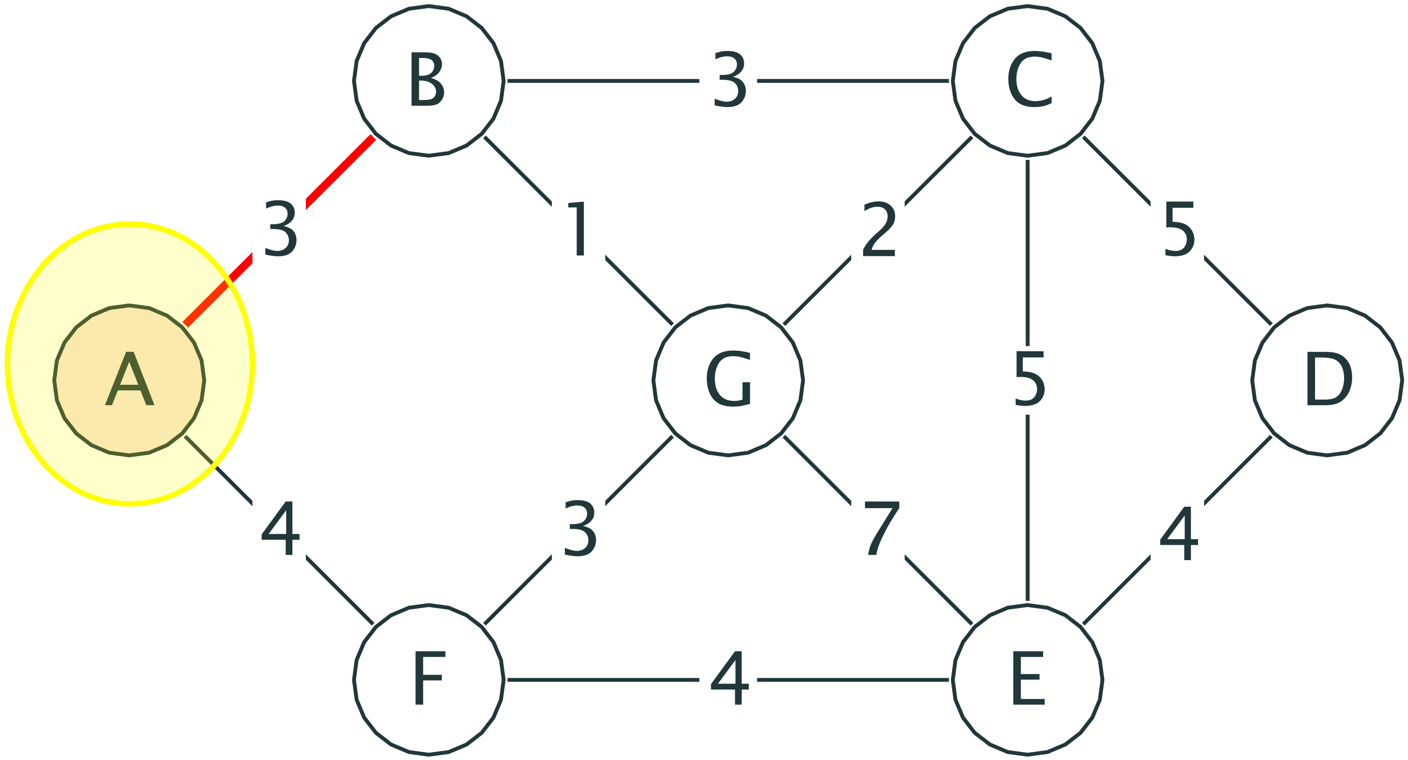
B. (GB), (GC), (AB), (GF), (CD), (DE)

C. (ED), (DC), (CG), (GB), (BA), (GF)

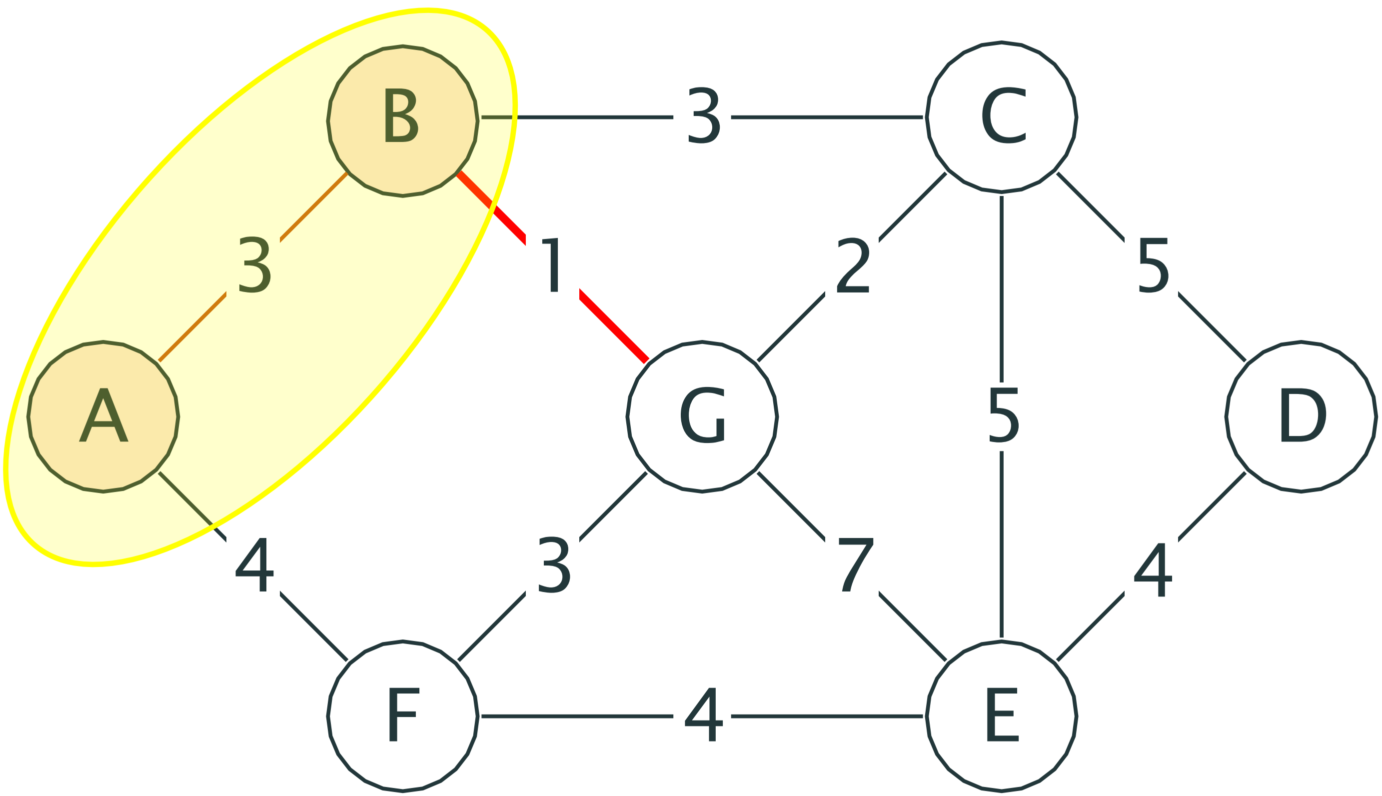
D. More than one of the above

E. None of the above

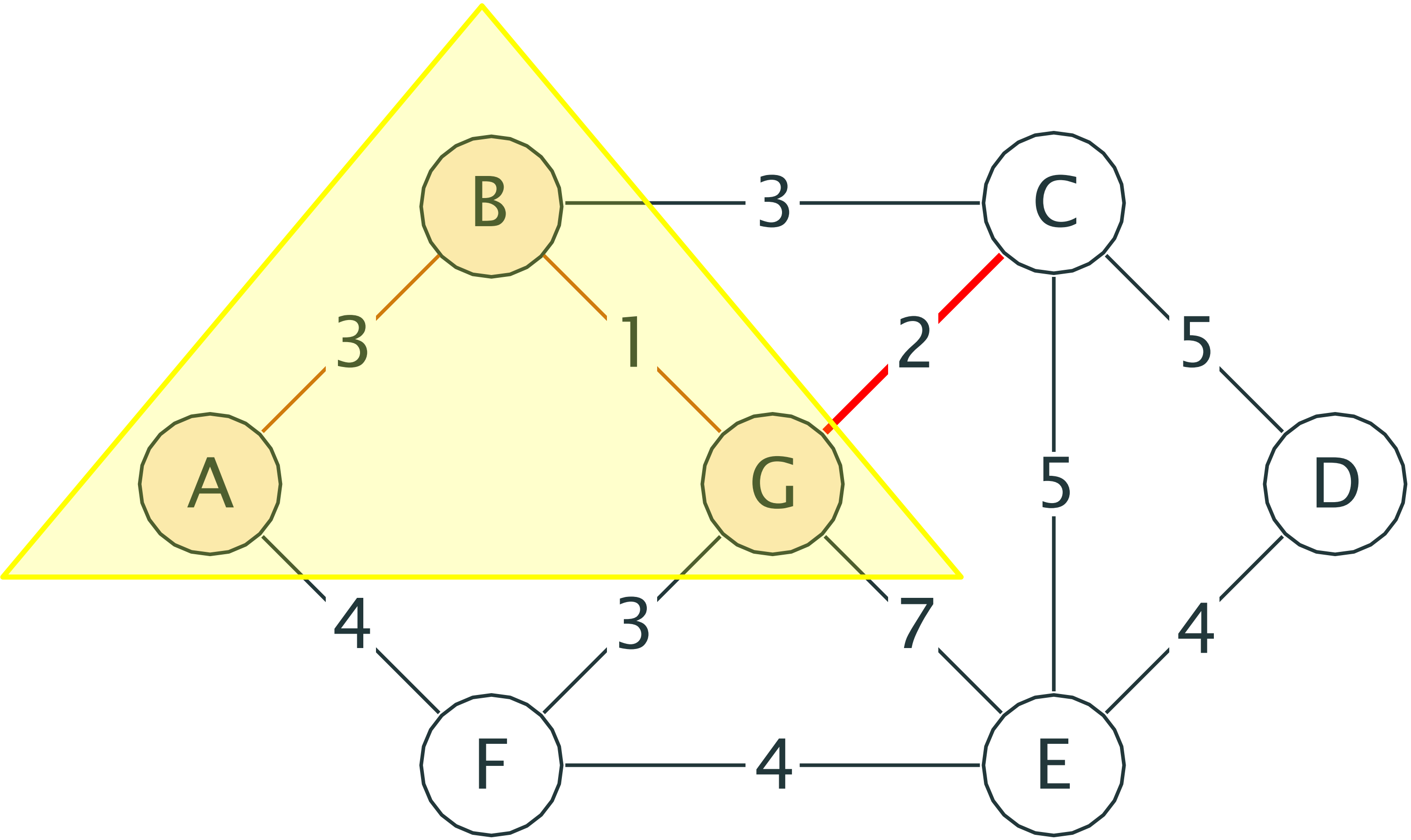
Prim: illustration



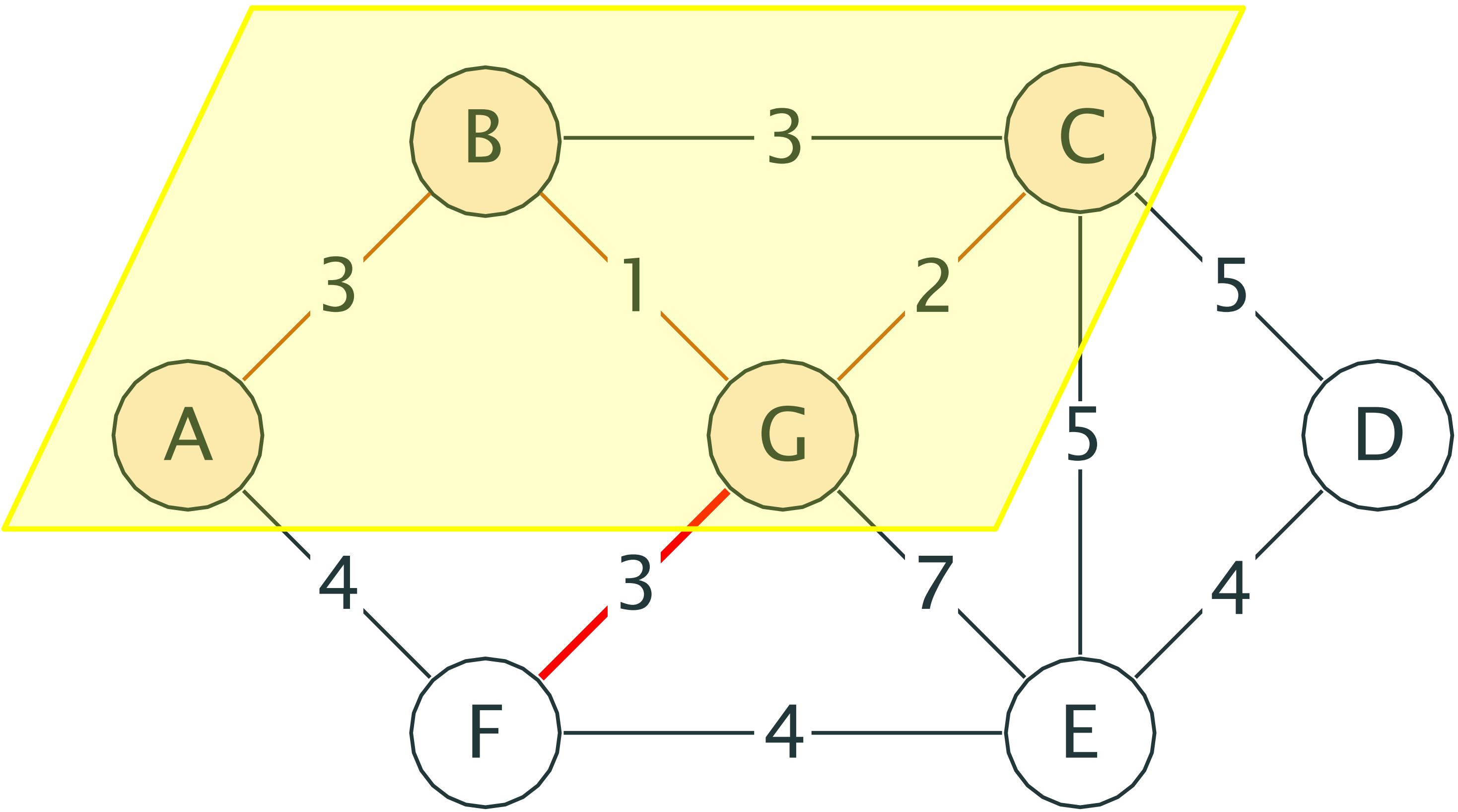
Prim: illustration



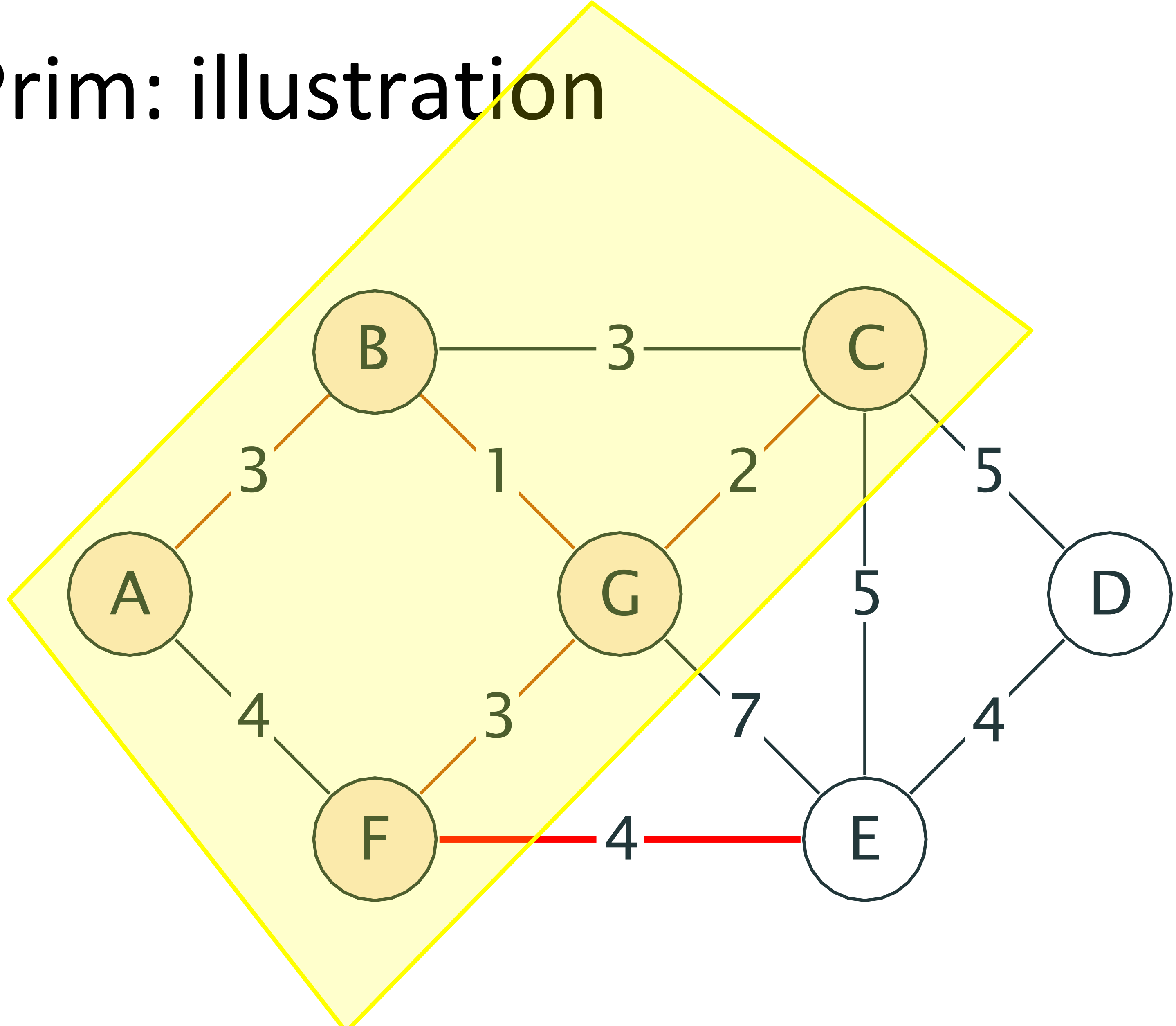
Prim: illustration



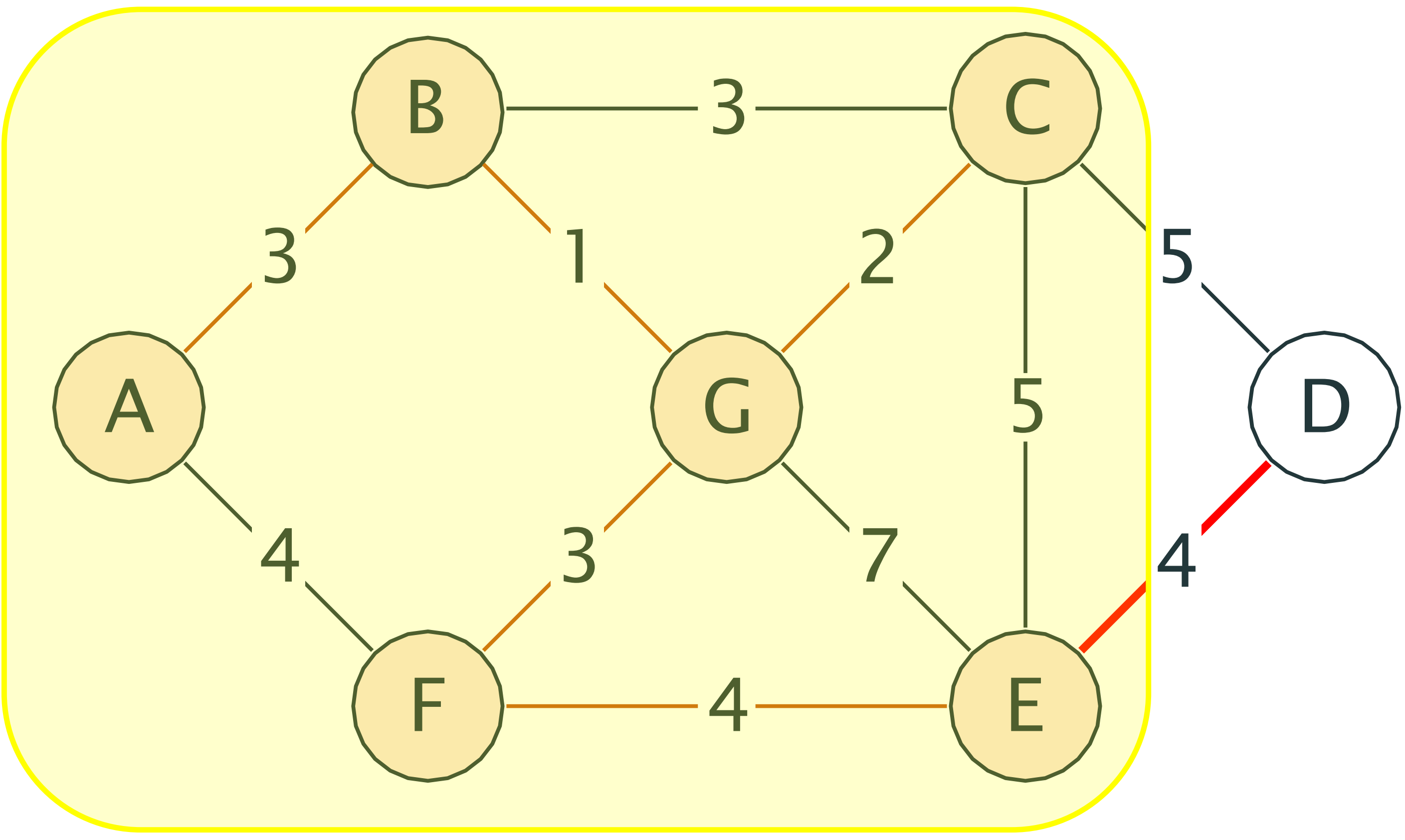
Prim: illustration



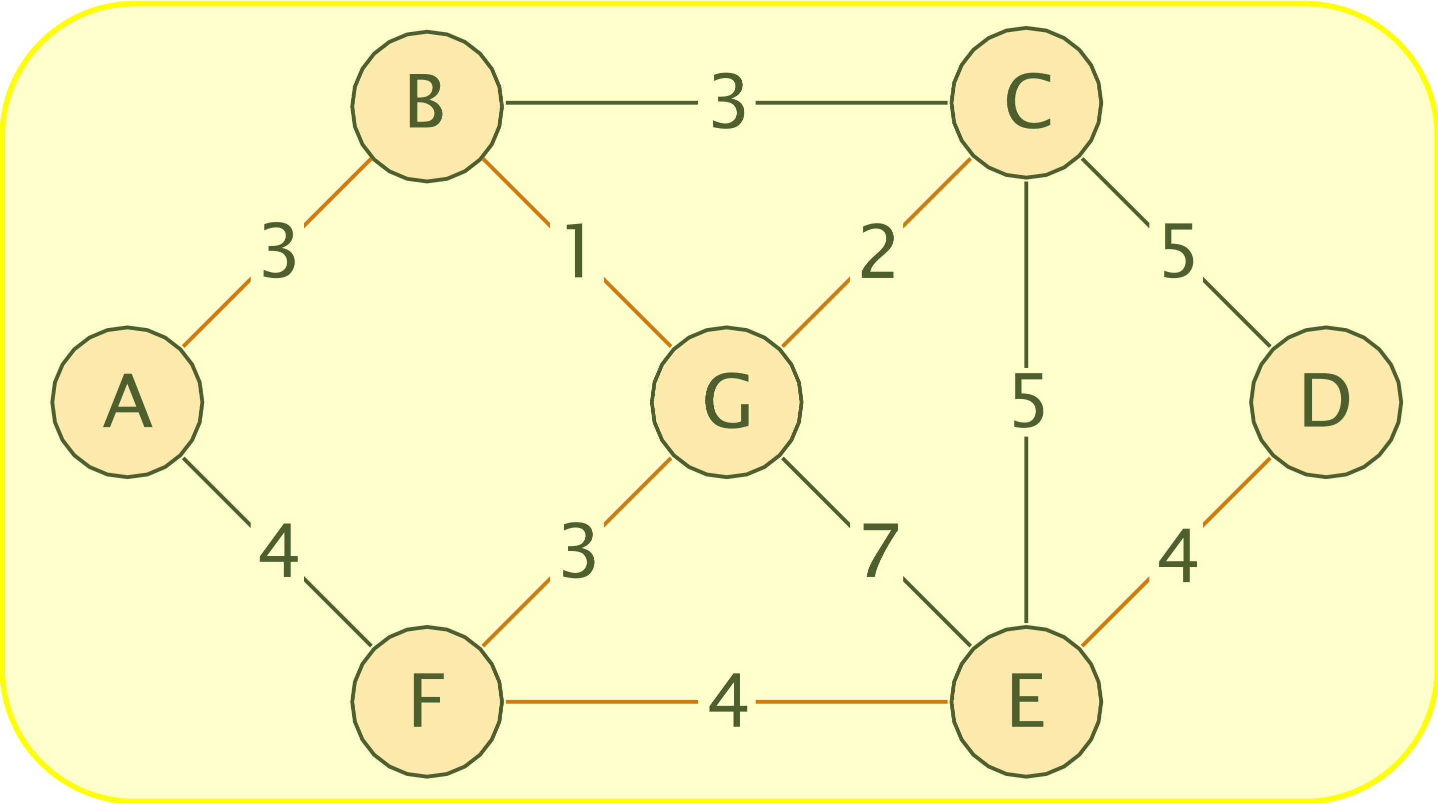
Prim: illustration



Prim: illustration



Prim: illustration



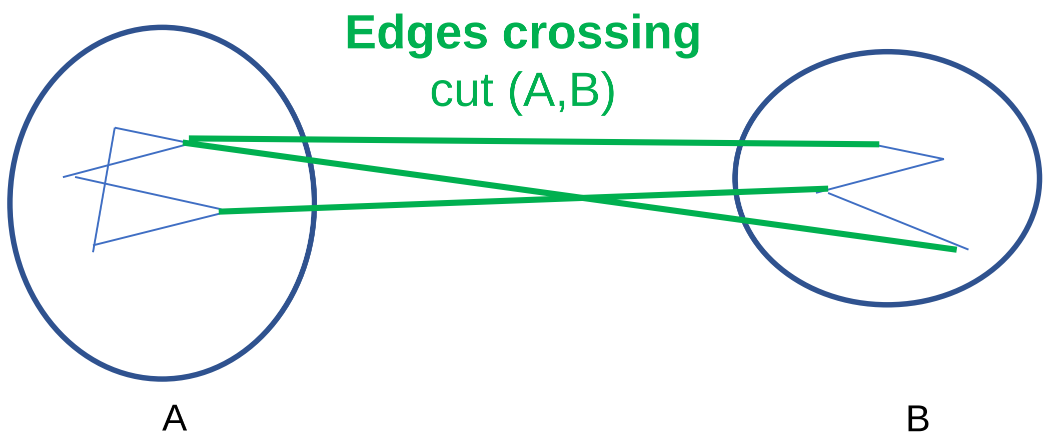
MST cost: 3 + 1 + 2 + 3 + 4 + 4 = 17

Prim’s Algorithm

- Prim always finds a minimum-cost spanning tree for any connected graph (even if the weights are negative)!
- How can we argue that Prim’s algorithm is optimal?
- Why is it always a good idea to take the cheapest edge from the existing tree-so-far?

Cuts

- A *cut* is a partition (A, B) of G into 2 non-empty subsets (proper subsets)
- How many different cuts can be in a G with n vertices? (n, n^2, 2^n)? $2^n - 2$

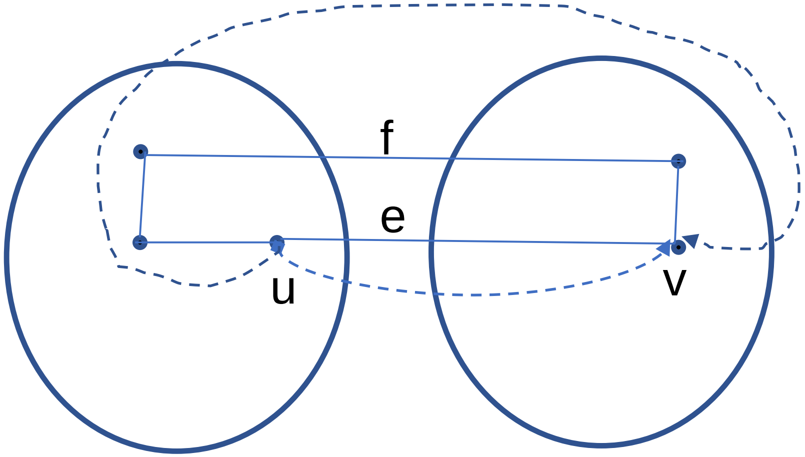


Crossing Edges Lemma

If there are (at least) two crossing edges for a cut (A,B) in an undirected connected graph, then these edges must be a part of some cycle.

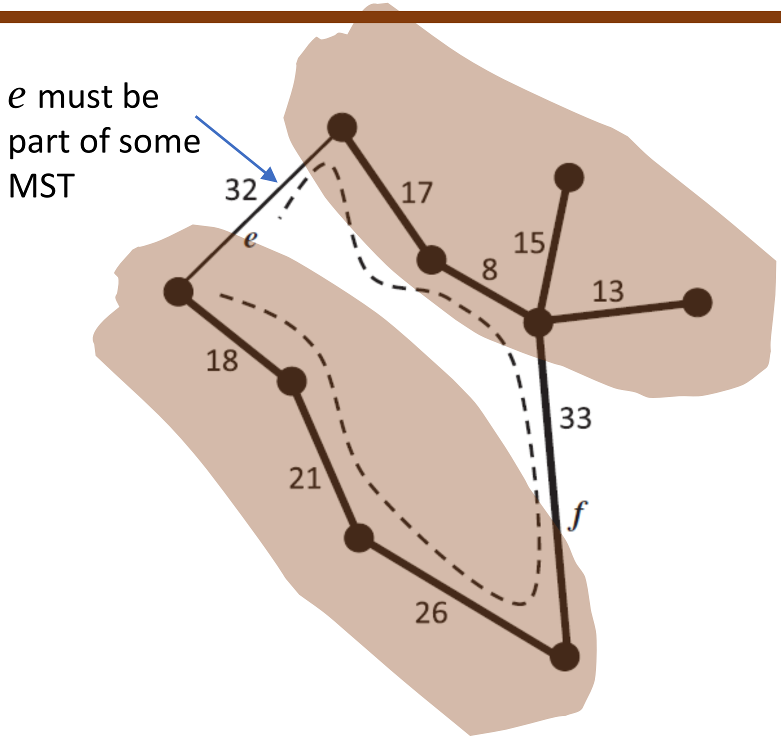
Proof

If there is a path from u to v from to two different partitions that includes the first crossing edge e, then the second crossing edge f offers an alternative path from u to v, thus closing the cycle on vertex v.



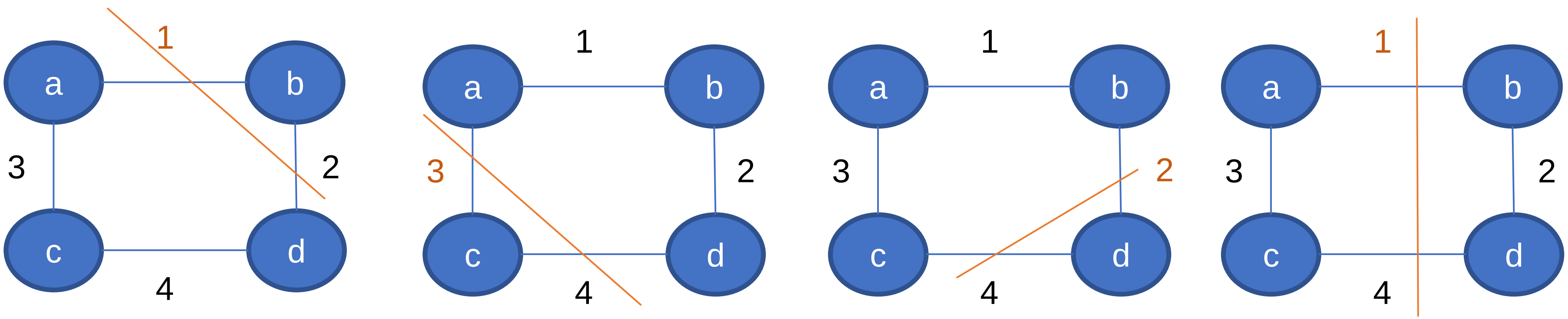
Cut Crossing Theorem

- Let G be a weighted connected graph, and let (A, B) be some cut of G.
- If *e* is the cheapest edge crossing cut (A, B), then *e* must be a part of some MST



What we are proving

If we have an edge in a graph and you can find just a single cut for which this edge has the min cost among all edges crossing this cut, then this edge **must** belong to the MST (or one of MSTs in case when the weights are not unique)



Cut 1
Edge 1 must be in MST

Cut 2
Edge 3 must be in MST

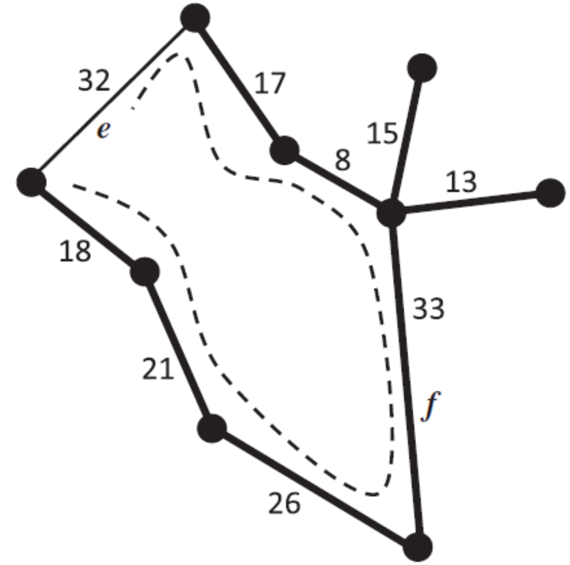
Cut 3
Edge 2 must be in MST

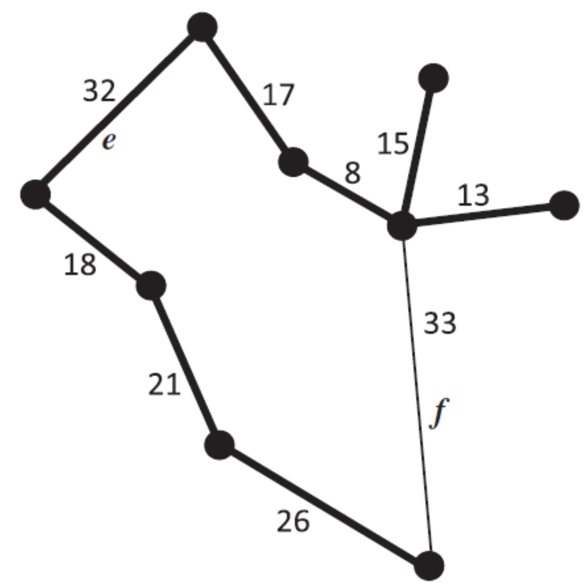
Cut 4
Edge 1 must be in MST

Note that edge 4 is never min of all crossing edges, no matter how we cut – so edge 4 is not in MST

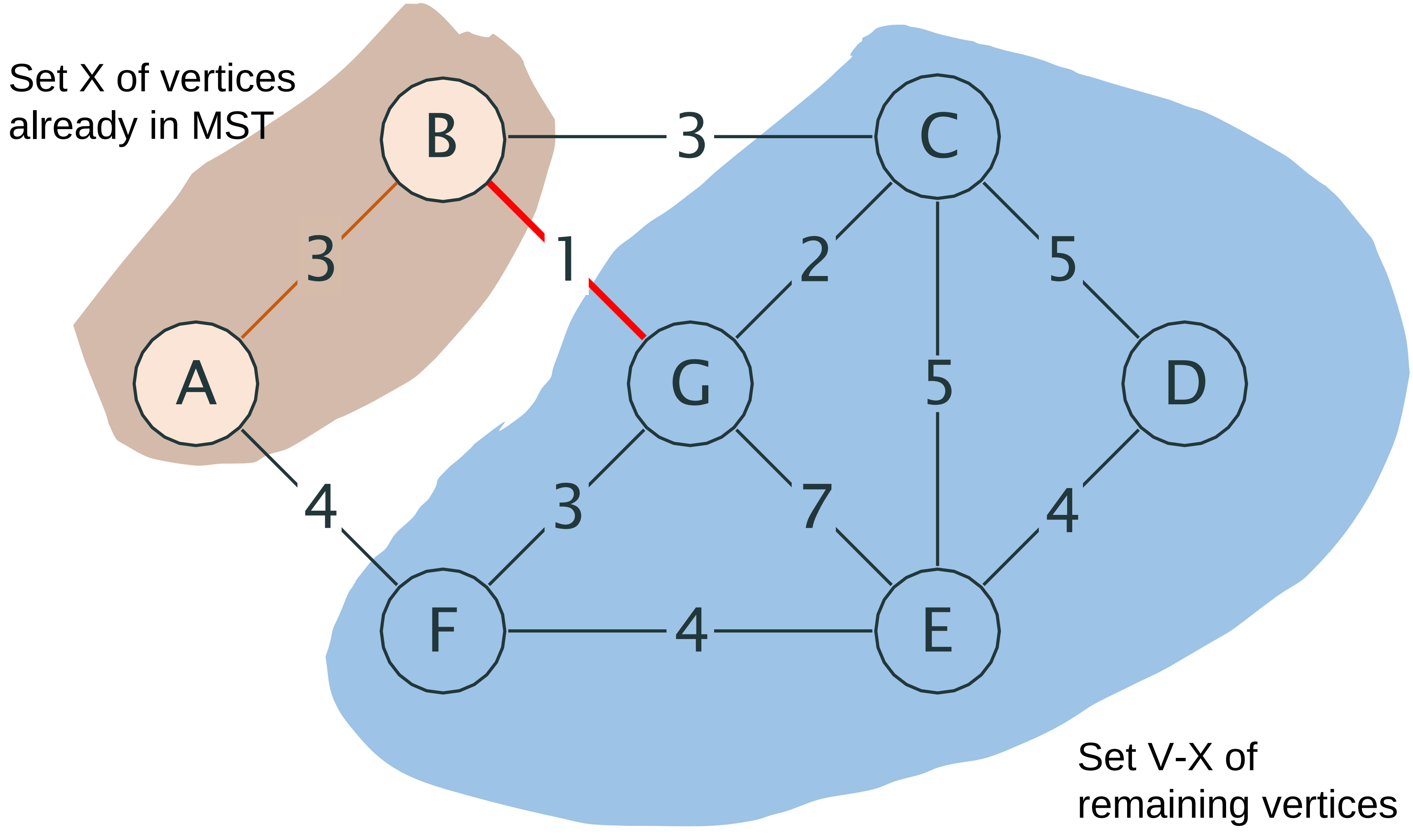
Exchange argument!

- Any nontree edge must have weight that is ≥ every edge in the cycle created by that edge and a minimum spanning tree.
- Suppose edge e has weight 32 and edge f in the same cycle has weight 33. Edge f is a part of MST (shown with bold edges), and edge e is not.
- But then we could replace f by e and get a spanning tree with lower total weight, which would contradict the fact that we started with a minimum spanning tree.





Prim: cut


Set X of vertices
already in MST
B
3
C
3
1
2
5
A
G
5
D
4
3
7
4
F
4
E
Set V-X of
remaining vertices

Algorithm Prim_MST (graph G(V,E))

```
initialize tree T: = ∅                # set of tree edges
X: = {vertex s}                       # s ∈ V, chosen arbitrarily
# X contains vertices spanned by the tree-so-far
Min-PQ: = ∅                 # set of all edges out of X prioritized by cost
current:= vertex s
while |X|!=|V|:
    for each e in neighbors(current):
        Min-PQ.enqueue(e)
    Select e =(u,v) as Min-PQ.dequeue()
    if u ∈ X and v ∉ X:
        add e to T
        add v to X
        current:=v
```

What data structure to use to check this quickly?

Prim: running time

Algorithm Prim_MST (graph G(V,E))

```
initialize tree T: = Ø
X: = {vertex s}
# X contains vertices spanned by the tree-so-far
Min-PQ: = Ø
current:= vertex s
while |X|!=|V|:
    for each e in neighbors(current):
         Min-PQ.enqueue(e)
    Select e =(u,v) as Min-PQ.dequeue()
    if u ∈ X and v ∉ X:
        add e to T
        add v to X
        current:=v
```

O(n) [while |X|!=|V|]

No more than O(m) edges in total, O(log m) for dequeue [for each e in neighbors(current)]

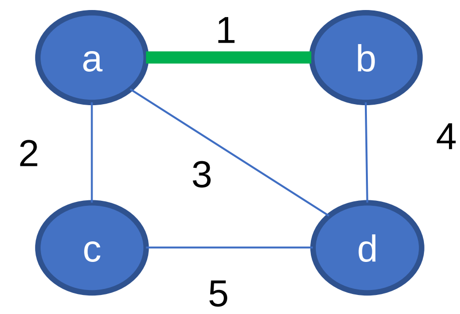
O(1) [if u ∈ X and v ∉ X]

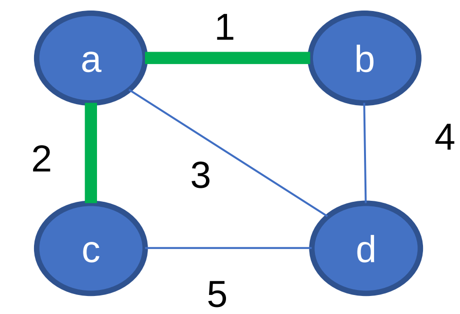
Total running time is O(m log m)

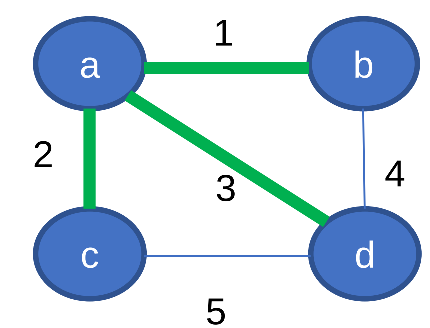
Algorithm by Kruskal

Sort all edges by weight (from smaller to larger – ascending)

Add the next smallest edge to the spanning tree, but only if adding it does not create a cycle







Sorted edges:

(a,b) ✓

(a,c) ✓

(a,d) ✓

(b,d) 🚫

(c,d) 🚫

Algorithm Kruskal_MST (graph G(V,E))

E_{sorted} := edges of G sorted by weights

T : = Ø   # set of spanning tree ***edges***

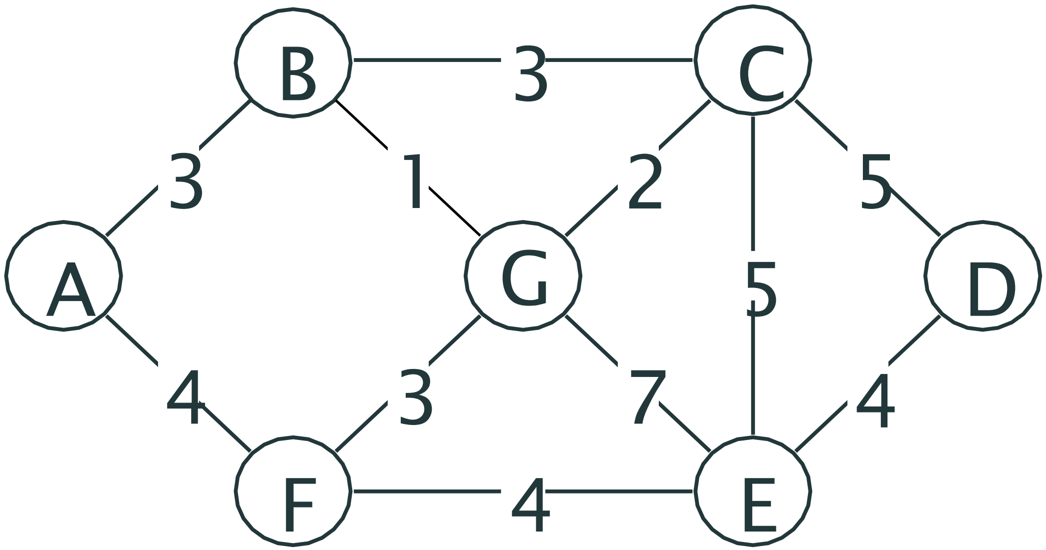
for i from 1 to m:

  if T U {E_{sorted}[i]} has no cycles

    add E_{sorted}[i] to T

return T

Which sequence represents the order in which edges are added to MST by the Kruskal algorithm?



A. (BG), (GC), (BA), (FG), (ED), (FE)

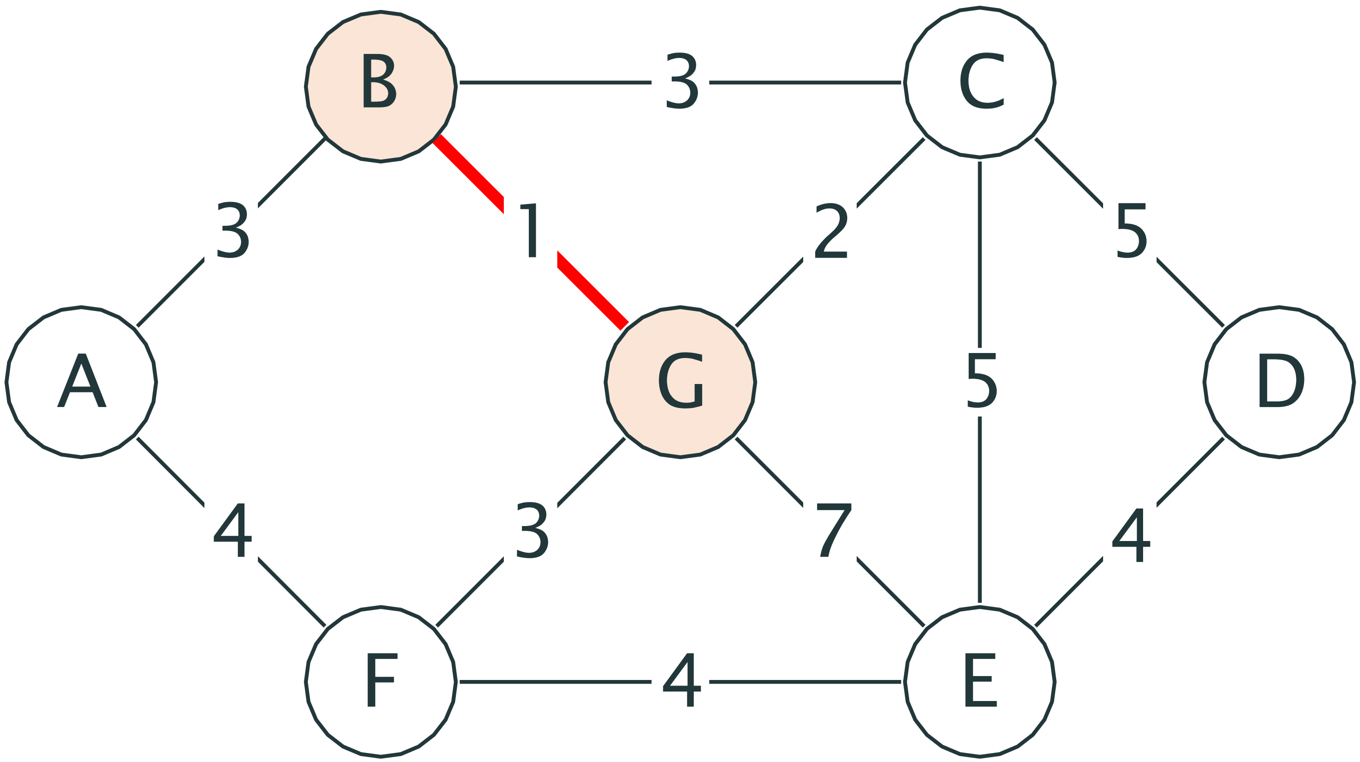
B. (BG), (BC), (BA), (CG), (ED), (FE)

C. (BG), (GC), (GF), (FE), (ED), (CE)

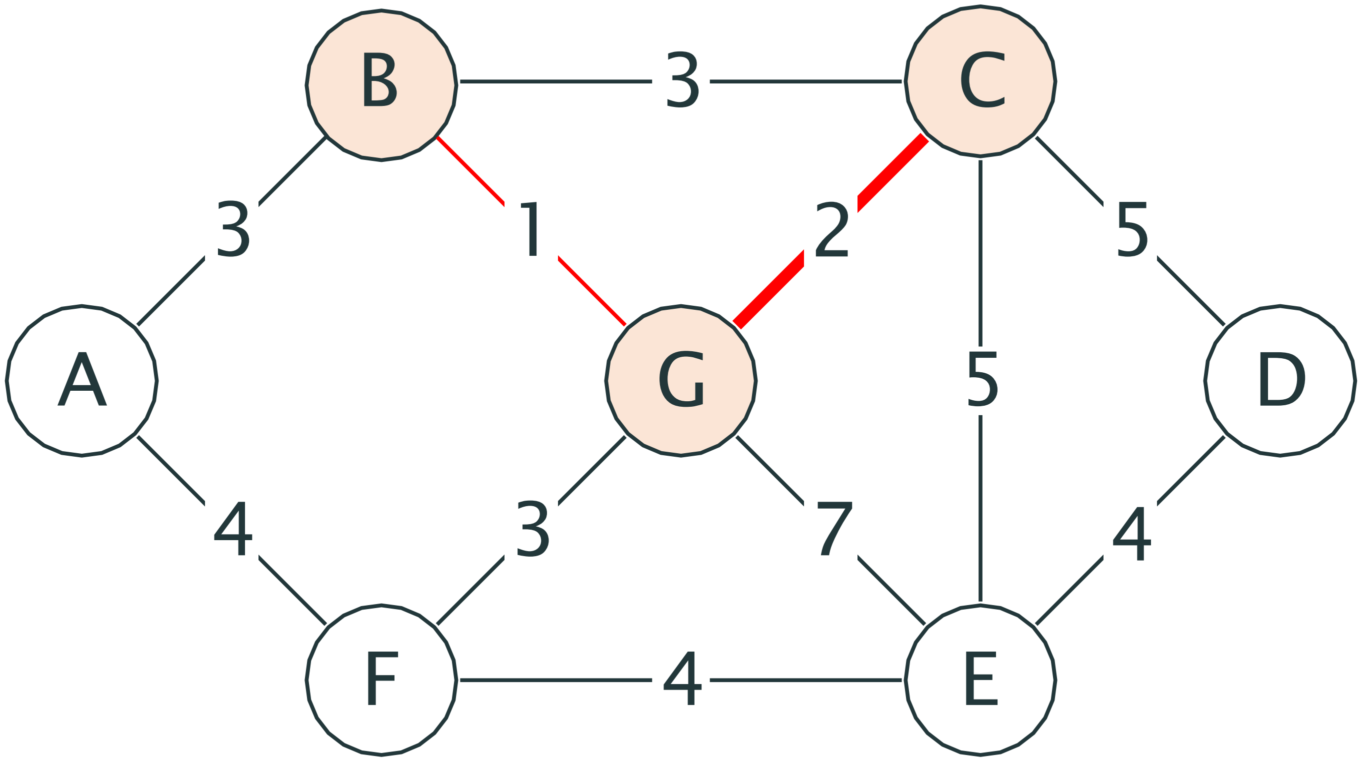
D. More than one of the above

E. None of the above

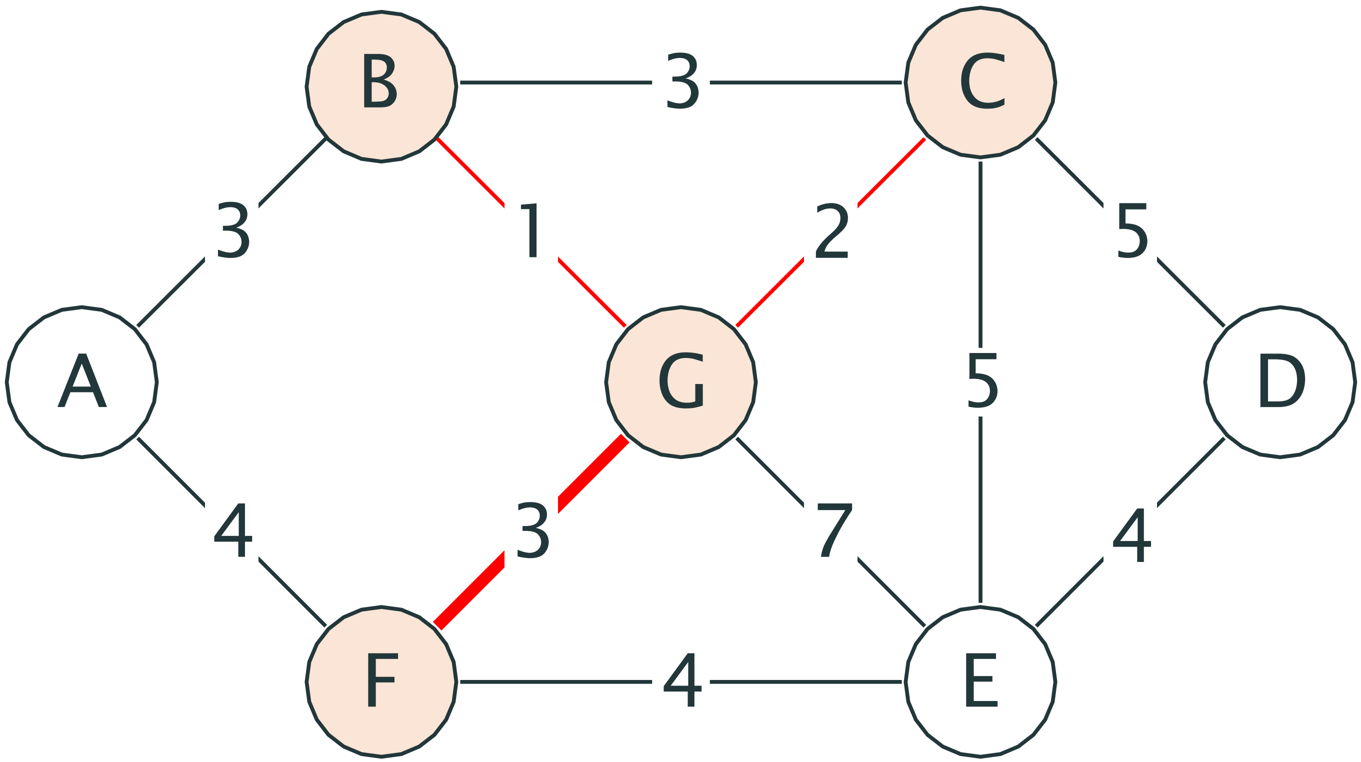
Kruskal illustration



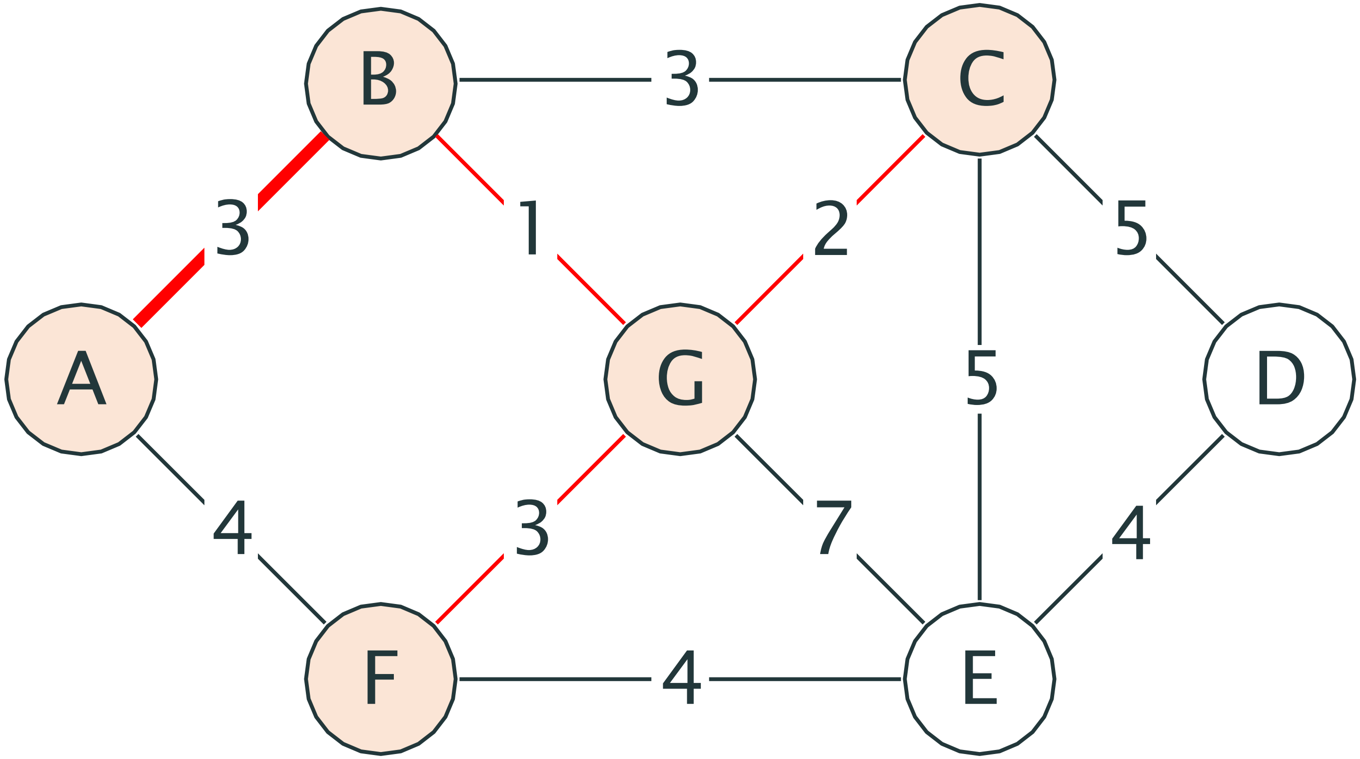
Kruskal illustration



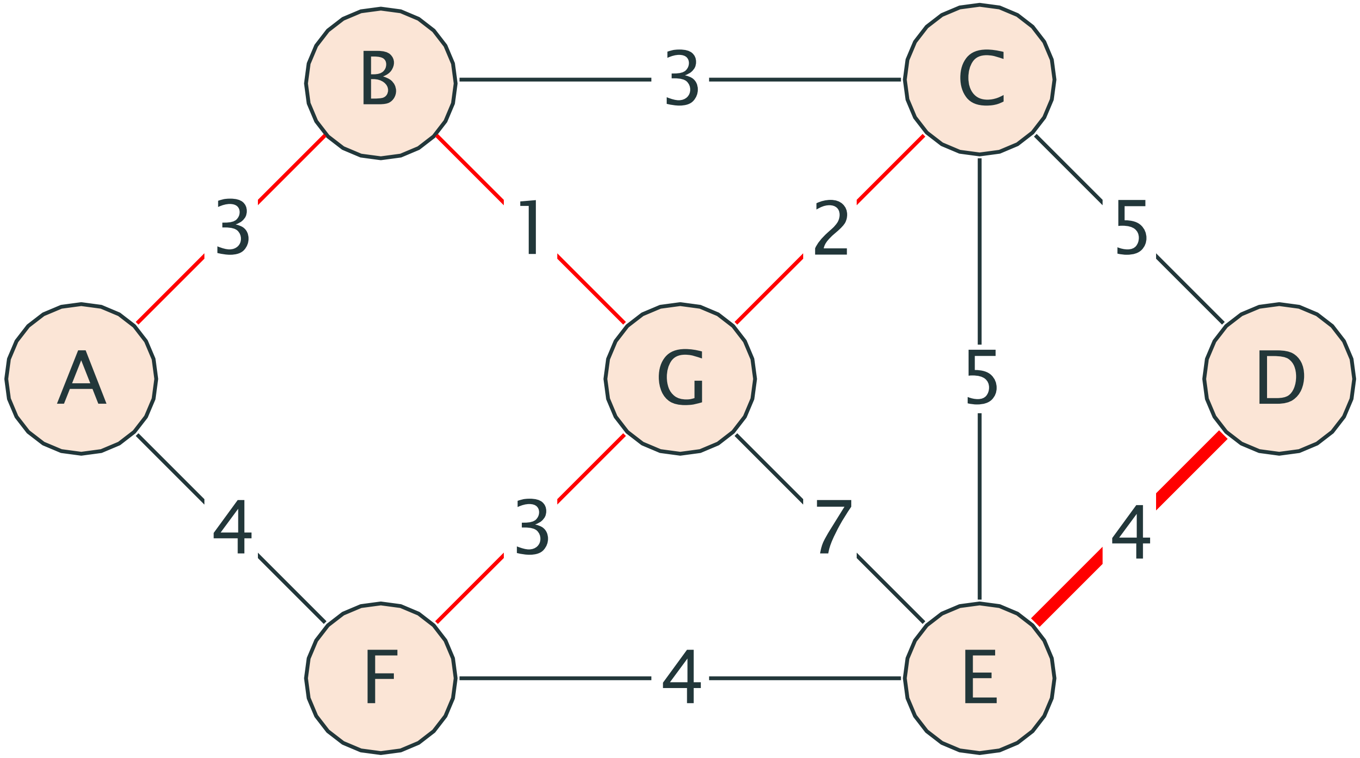
Kruskal illustration



Kruskal illustration

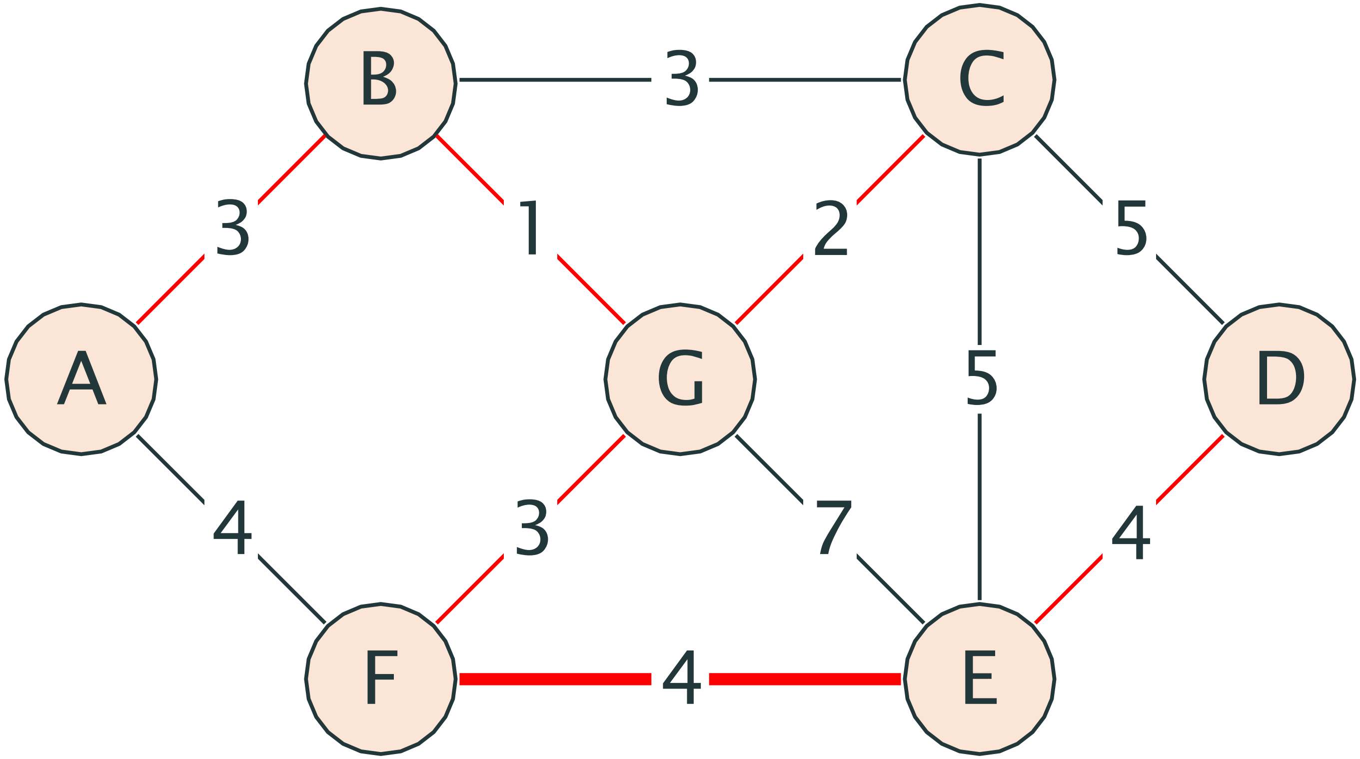


Kruskal illustration



Note that at this point T is not even a spanning tree
(not connected)

Kruskal illustration



MST cost: 1 + 2 + 3 + 3 + 4 + 4 = 17

Kruskal algorithm

Algorithm Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø                                   # collects edges of the future MST

for i from 1 to m:
    if T U {E'[i]} has no cycles
        add E'[i] to T

return T
```

Repeatedly add a minimum-cost edge
that does not create a cycle

Kruskal algorithm

Algorithm Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø                            # collects edges of the future MST

for i from 1 to m:
    if T U {E'[i]} has no cycles
        add E'[i] to T

    if |T| = |V| - 1:              # we can stop once we have a tree
        break

return T
```

Stop when
n-1 edges have been selected

Running time

Kruskal_MST (graph G(V,E))

```
1  E' := edges of G sorted by weights
2  T : = Ø

3  for i from 1 to m:
4          if T U {E'[i]} has no cycles
5                  add E'[i] to T

6          if |T| = |V| - 1:
7                  break

8  return T
```

Line 1: sorting m edges by weight. O(m log m). This is the same as O(m log n) Why?

Line 3: outer *for* loop. O(m). We check all m edges in the worst case.
Line 4: need to find if edge E'[i]= (u,v) creates a cycle.
Find out if there is already a path from u to v in T by any graph traversal (DFS or BFS). DFS of T with n vertices and n-1 edges is O(n + n) = O(n).

Thus, total time of the for loop is O(m)*O(n) = O(mn)

$O(n^3)$ for dense graphs

Kruskal MST runs in time O(m log n) + O(mn) = **O (mn)**

Running time

Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø

for i from 1 to m:
        if T U {E'[i]} has no cycles
                add E'[i] to T

        if |T| = |V| - 1:
                break

return T
```

Bottleneck: detecting a cycle

Kruskal MST runs in time **O (mn)**

Can we do better?

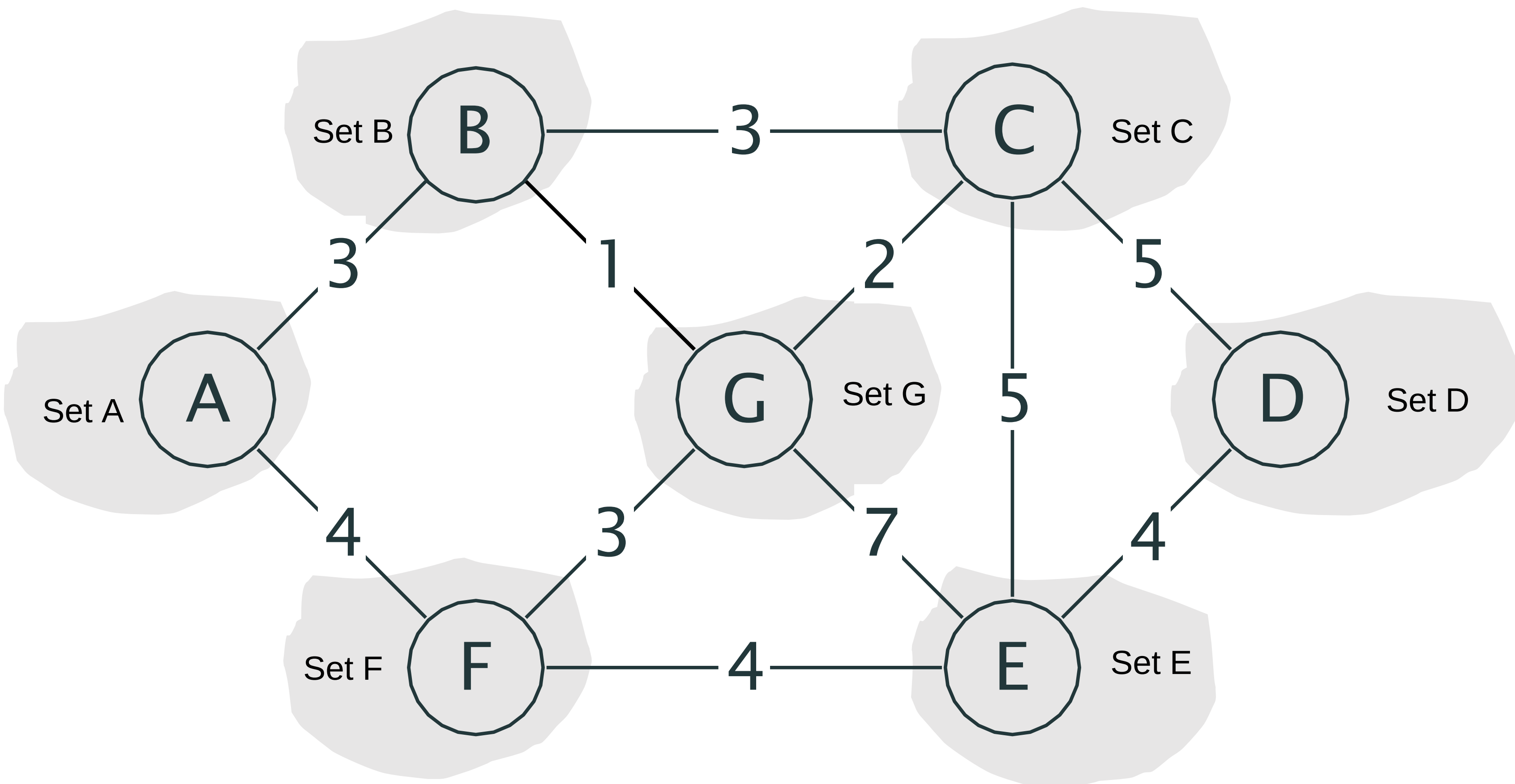
Kruskal as union of sets

We can look at Kruskal from a Set point of view

- First we have n sets: each vertex i is in its own set S_i – we need to be able to MAKE-SET for a single element
- Next we combine two sets of vertices S_i and S_j into one set: we perform UNION (S_i and S_j), adding an edge (u,v) such that $u \in S_i$ and $v \in S_j$
- However we do the union only if $S_i \neq S_j$. In other words, we need to know if u and v are already in the same set, in the same connected component, we need to FIND out set names for u and for v and compare them for equality

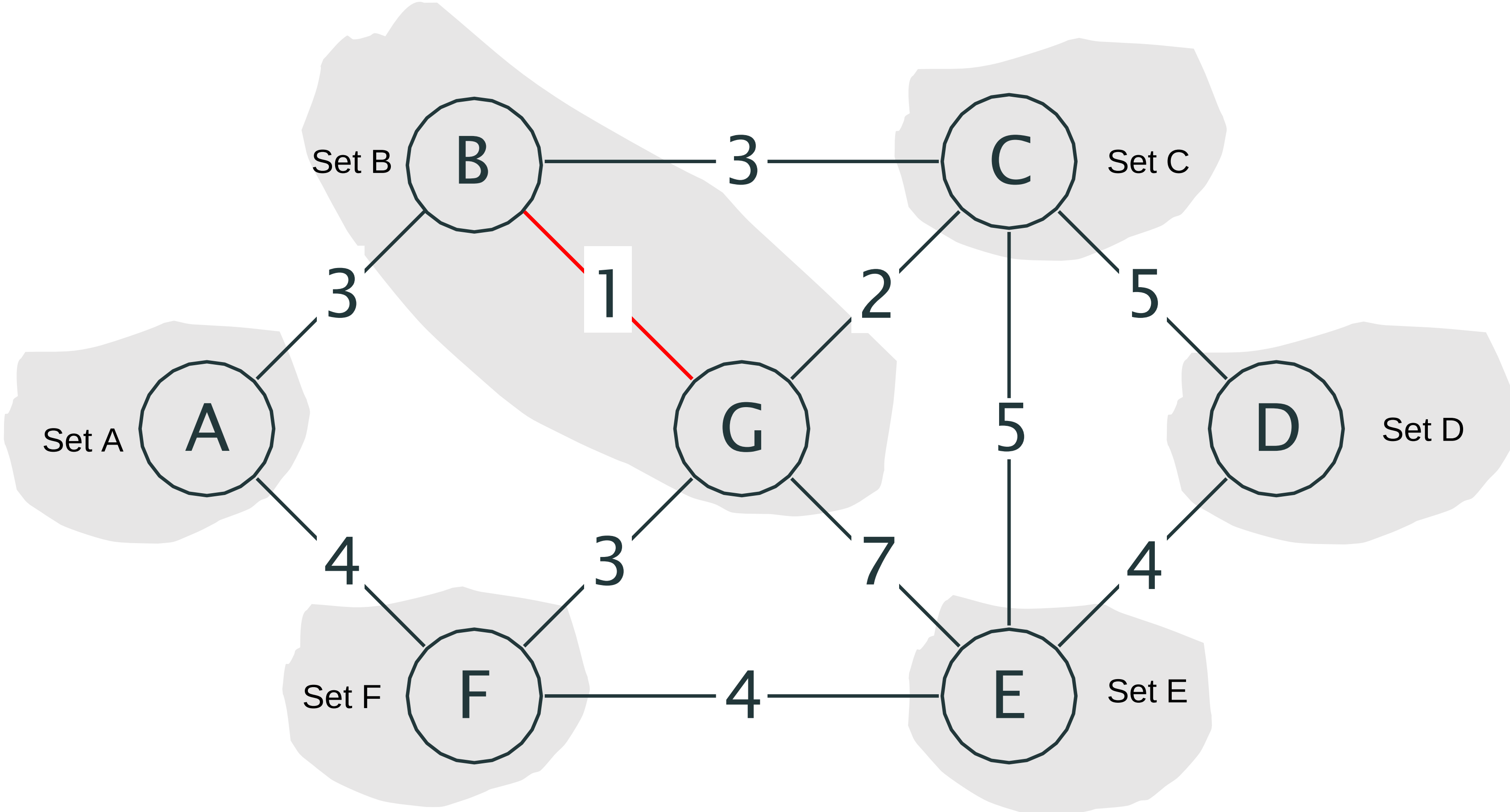
Note that all the sets are *disjoint*: each node belongs to a single set during the execution of the algorithm

Kruskal as union of sets



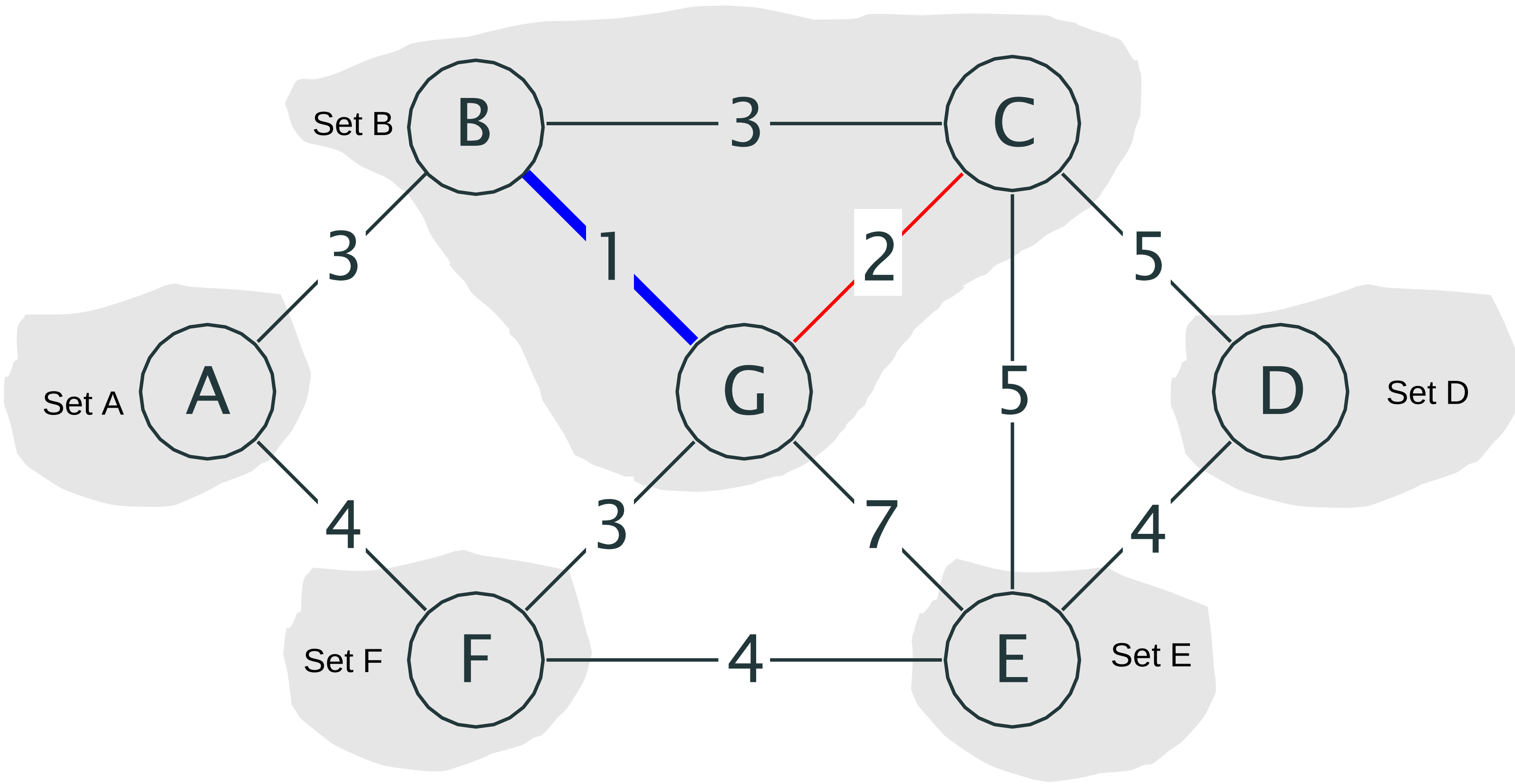
Set B = UNION (B, G)

Kruskal as union of sets



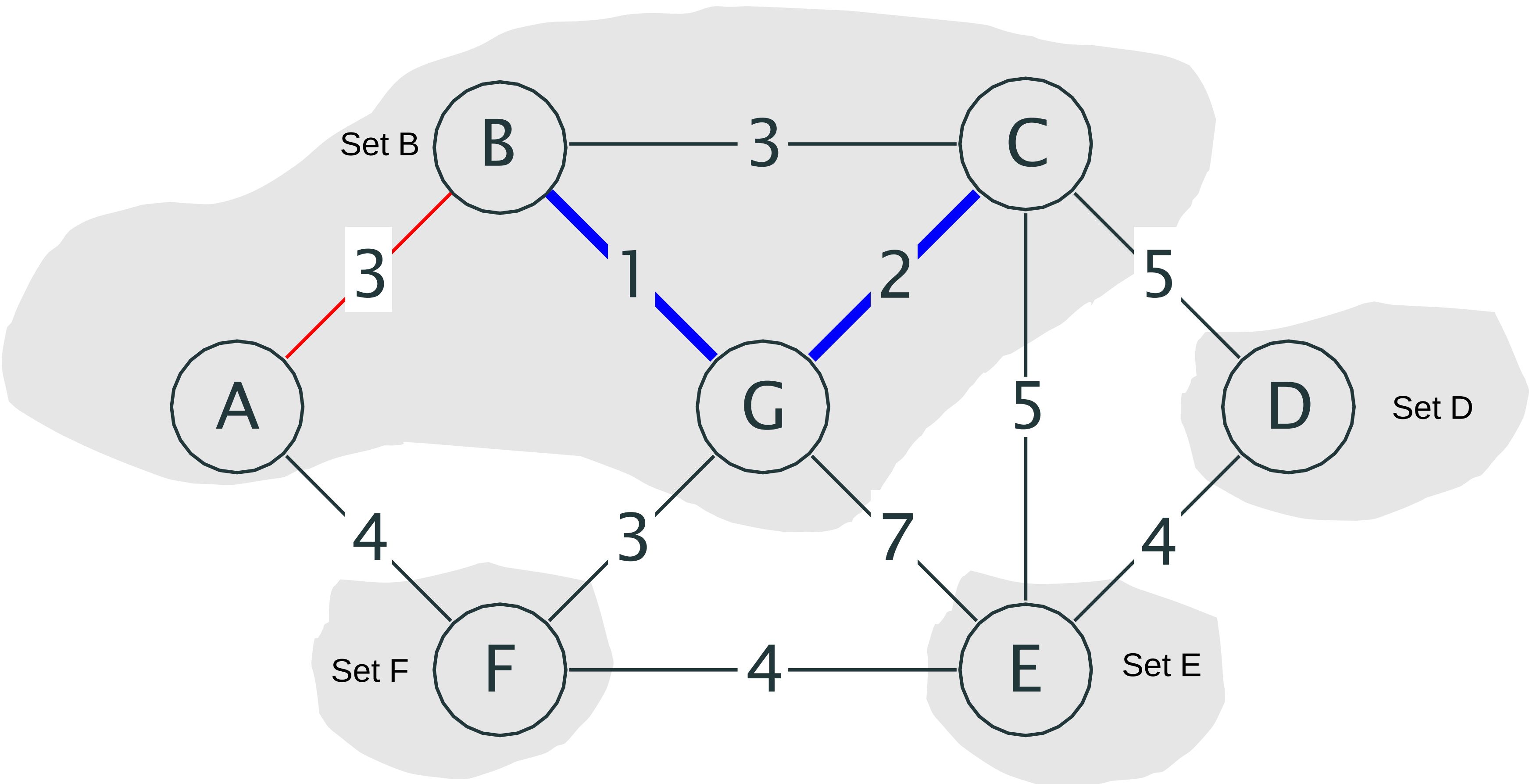
Set B = UNION (B, C)

Kruskal as union of sets



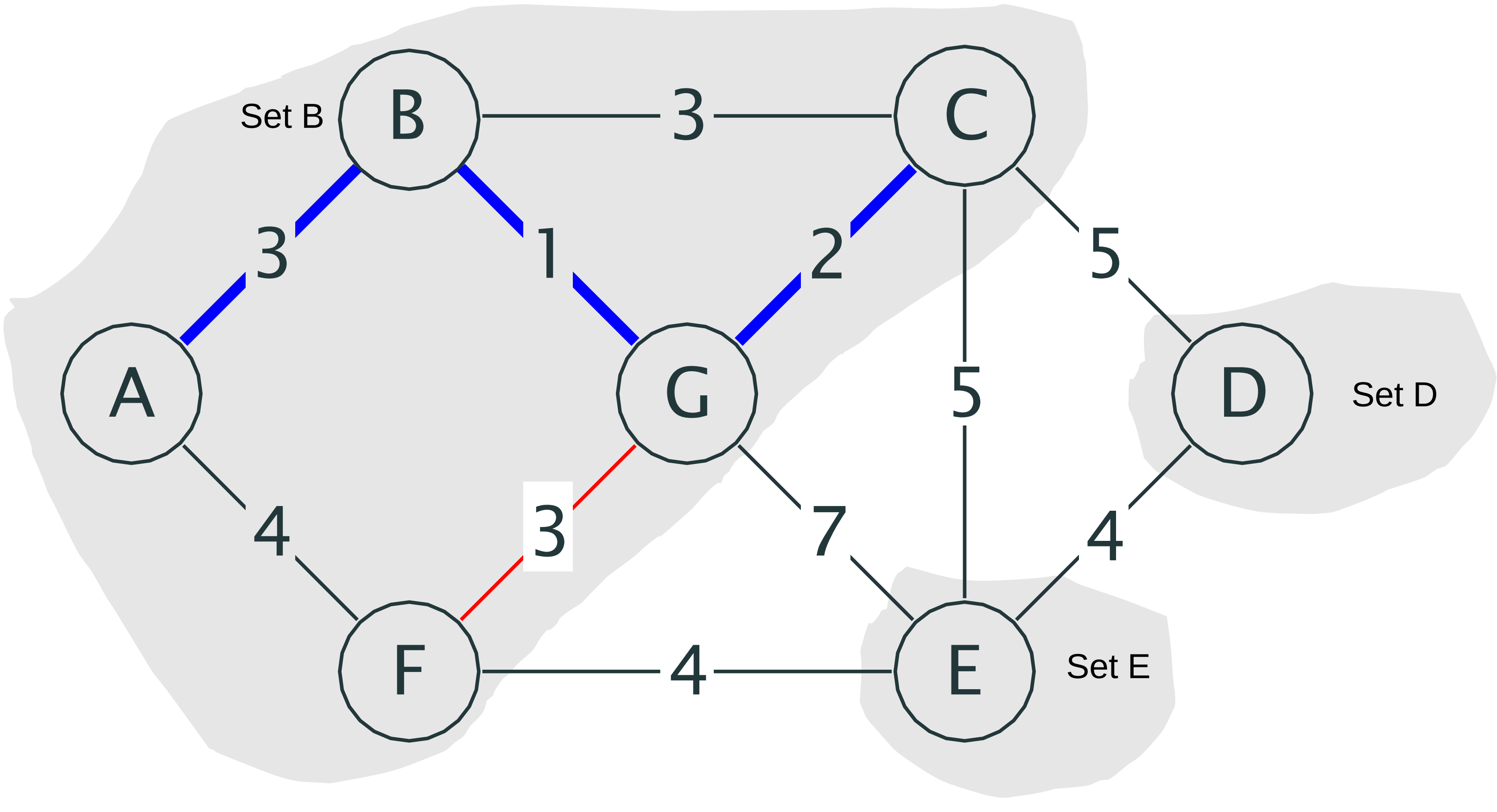
Set B = UNION (B, A)

Kruskal as union of sets



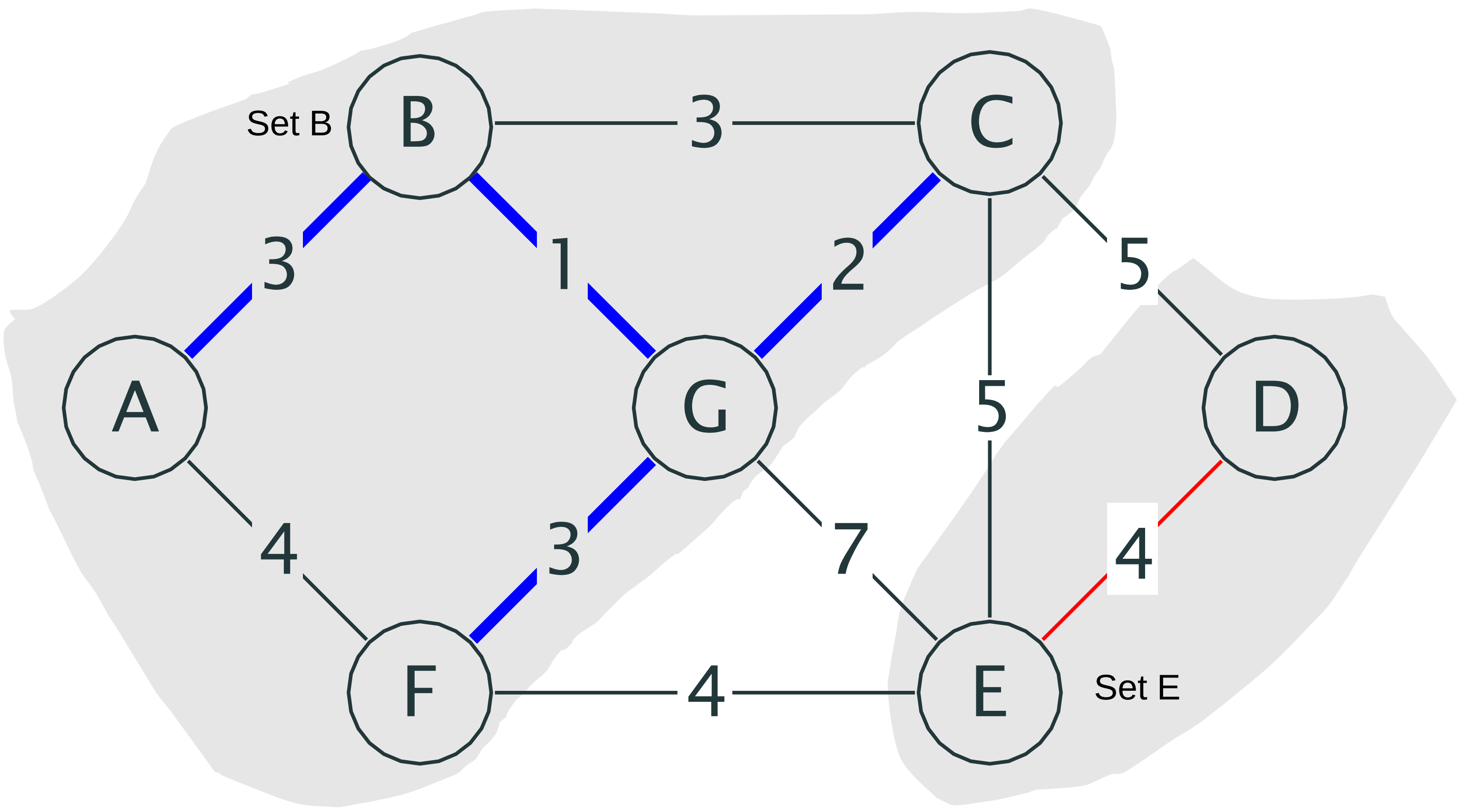
Set B = UNION (B, F)

Kruskal as union of sets



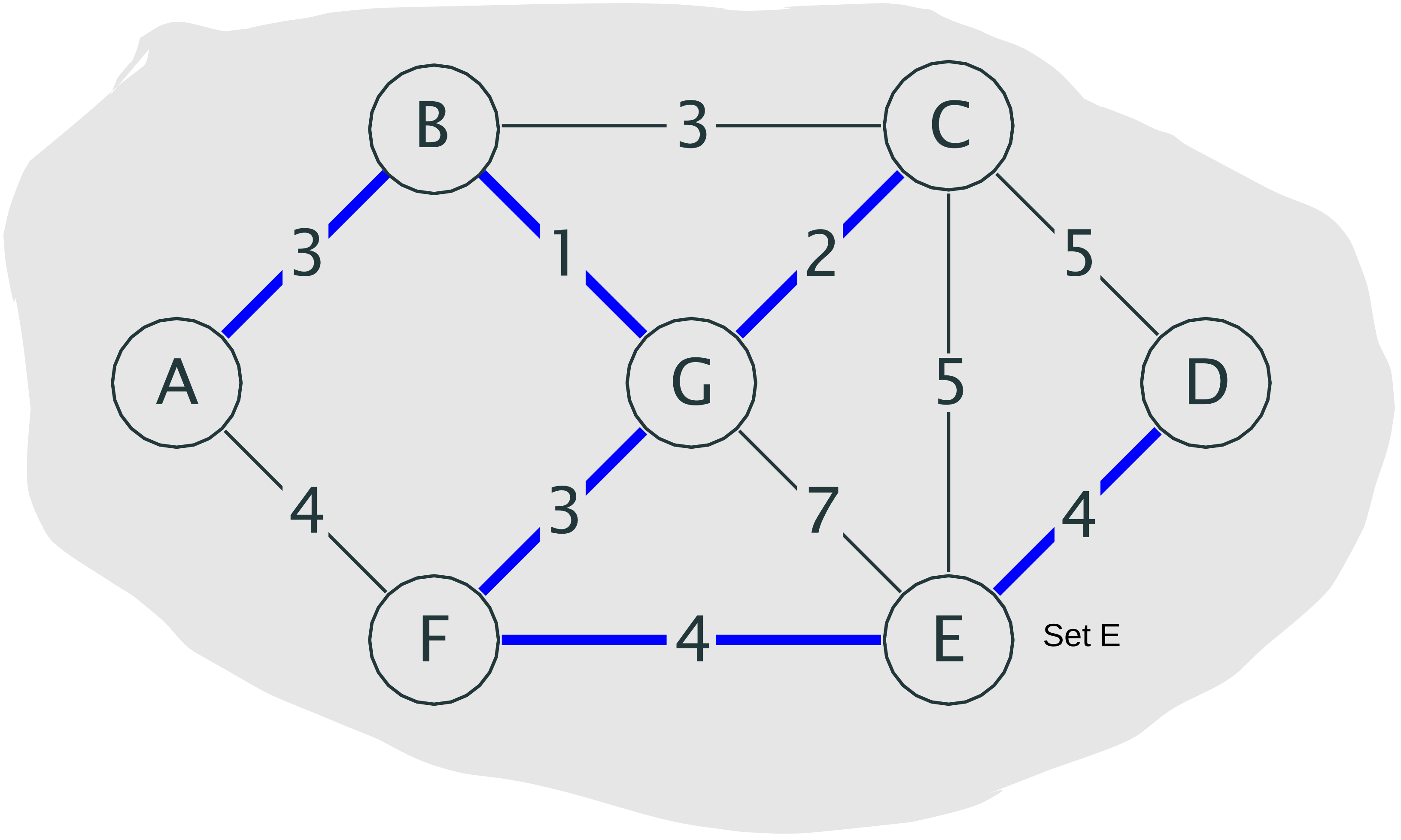
Set E = UNION (D, E)

Kruskal as union of sets



Set E = UNION (B, E)

Kruskal as union of sets



Set spanning all vertices of G with selected edges:
MST of G

New ADT: UNION-FIND (= Disjoint Set ADT)

UNION-FIND is an Abstract Data Type that supports the following operations:

- *MAKESET(x)*: Creates a new set X containing a single element x.
- *UNION(X, Y)*: Creates a new set containing the elements of sets X and Y in their union and deletes the previous sets X and Y.
- *FIND(x)*: Returns the name of the set to which element x belongs.

Read about an efficient implementation of UNION-FIND in more detailed slides or in this textbook chapter

Kruskal running time with UNION-FIND

Kruskal_MST (graph G(V,E))

```
1   E' := edges of G sorted by weights
2   T : = ∅

3   for  i from 1 to n:
4           MAKE-SET (node i)

5   for each edge (u,v) in E':
6           if FIND(u) ≠ FIND(v):
7                   T: = T U (u,v)
8                   UNION(u, v)

9           if |T| = |V| - 1:
                    break

    return T
```

Line 1: sorting m edges by weight. O(m log n).

Line 3: Making an array of size n: O(n).

Line 5: O(m) edges in the worst case.
For each edge: perform FIND: O(log n) and

sometimes UNION: in time O(1)

Thus, total time of the for loop is O(m log n)

Kruskal MST with UNION-FIND runs in time O(m log n) + O(n) + O(m log n) = **O (m log n)**

Both MST algorithms are greedy

All the algorithms follow some greedy strategy.

Algorithm MST (graph G(V,E))

```
T: = Ø                # collects edges of the future MST

while |T| ≤ |V| - 1:    # needs to collect n-1 edges
    select next edge e from E        # some greedy move
    T: = T U e

return T
```