Many algorithms use Priority Queues

- **Dijkstra's algorithm**: finding a shortest path in a graph
- **Prim's algorithm**: constructing a minimum spanning tree of a graph
- **Huffman encoding**: constructing an optimum prefix-free encoding of a string
- **Heap sort**: sorting a given sequence

ADT and Data structures.
Heap Sort

[Review 02.03]

by Marina Barsky

We can sort using Heaps!

- After array elements are enqueued:
- Produce a sorted array by dequeuing them

Algorithm HeapSort

HeapSortNaive (array *A* of size *n*)

create an empty max-heap

for *i* from 0 to *n*-1:
    enqueue (*A*[*i*])

for *i* from *n*-1 downto 0:
    A[*i*] ← dequeue()

What is the running time?

Heapsort: naive

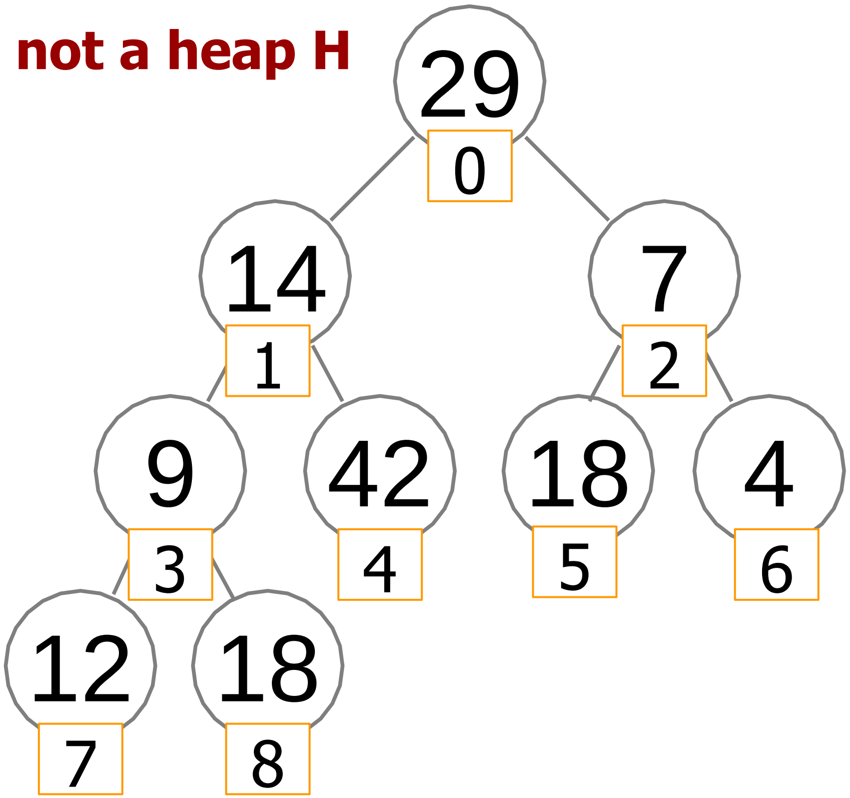
- The resulting algorithm has running time $O(n \log n)$
- Natural generalization of *selection sort*: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure
- Not in-place: uses additional space $O(n)$ to store the heap

In-place Heapsort: all is done inside the input array

- Turn input array A of size *n* into a heap of size *m*=n by rearranging its elements
- After this, extract max at A[0] and swap it with the element A[*m*-1]
- Decrement heap size *m*:= *m* -1
- Restore heap
- Continue until heap size m=1

How to Heapify an array

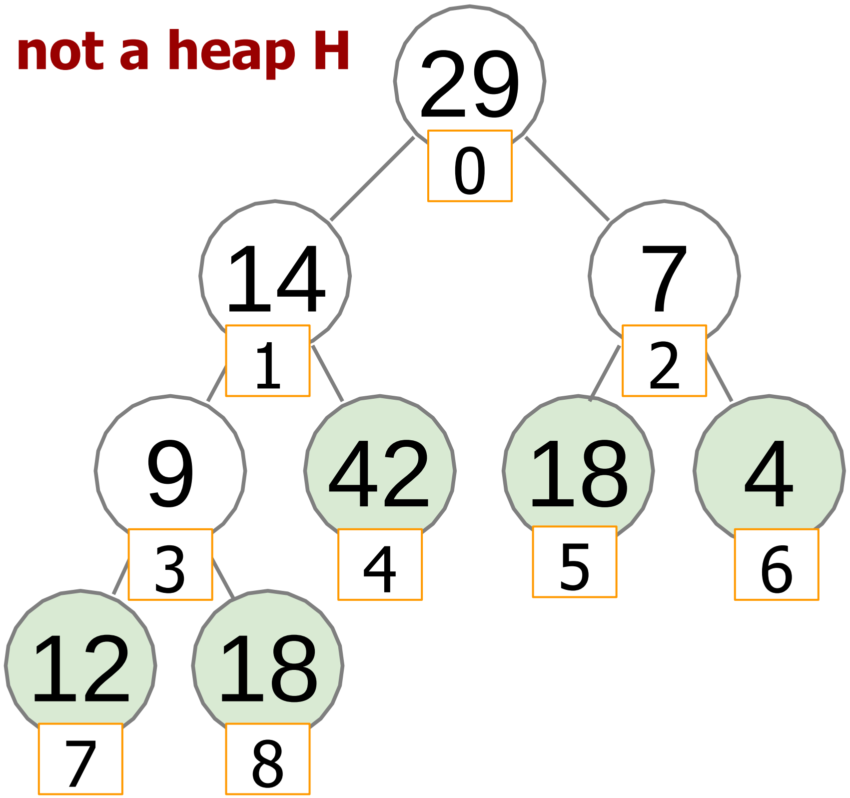
not a heap H



➔ Lets' go bottom up and repair heap property for all subtrees rooted at current node

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

How to Heapify an array

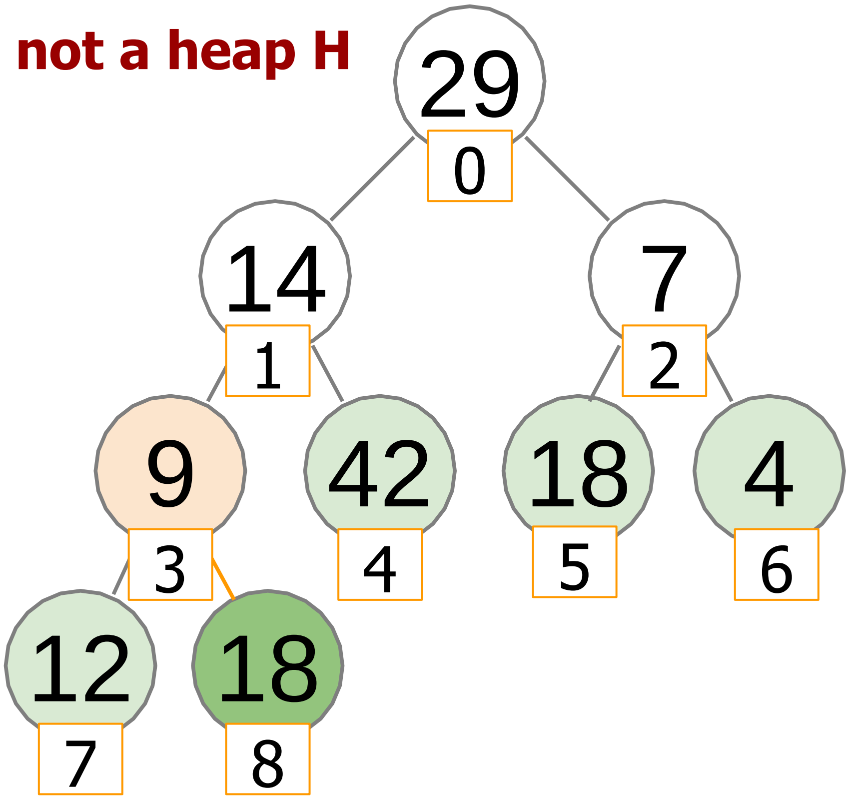


29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

- Lets' go bottom up and repair heap property for all subtrees rooted at current node
- If current node is a leaf, then it does not need to be repaired
- How do we find the first from the end node that is not a leaf?

How to Heapify an array

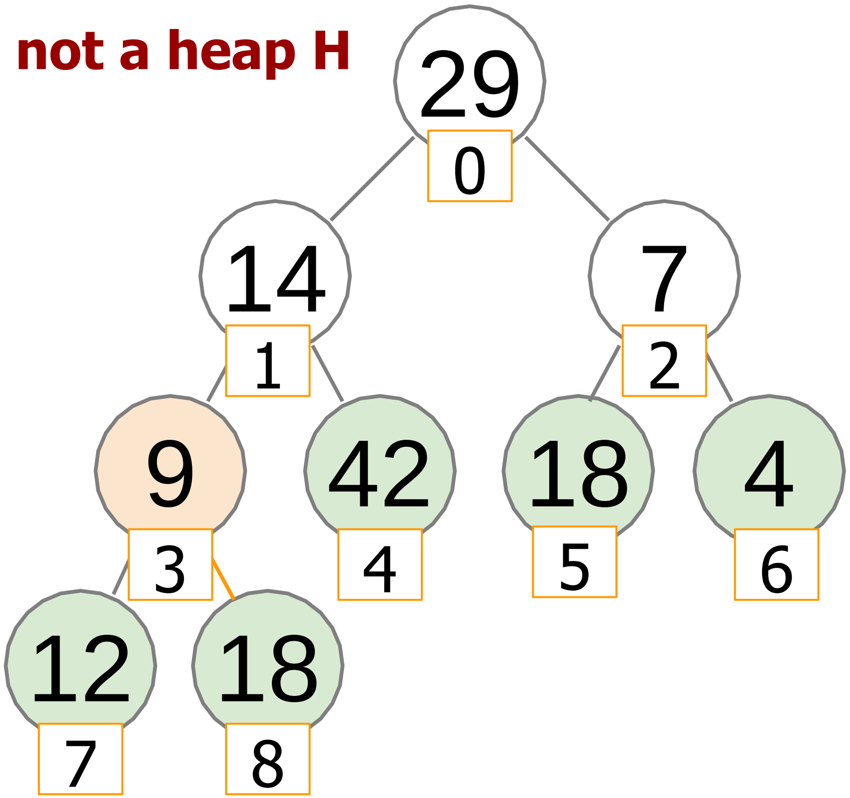
not a heap H



29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

- Lets' go bottom up and repair heap property for all subtrees rooted at current node
- If current node is a leaf, then it does not need to be repaired
- How do we find the first from the end node that is not a leaf?
  - We find the parent of the last leaf H[n - 1]:
  - parent(i) = $\lfloor (i - 1)/2 \rfloor$

How to Heapify an array

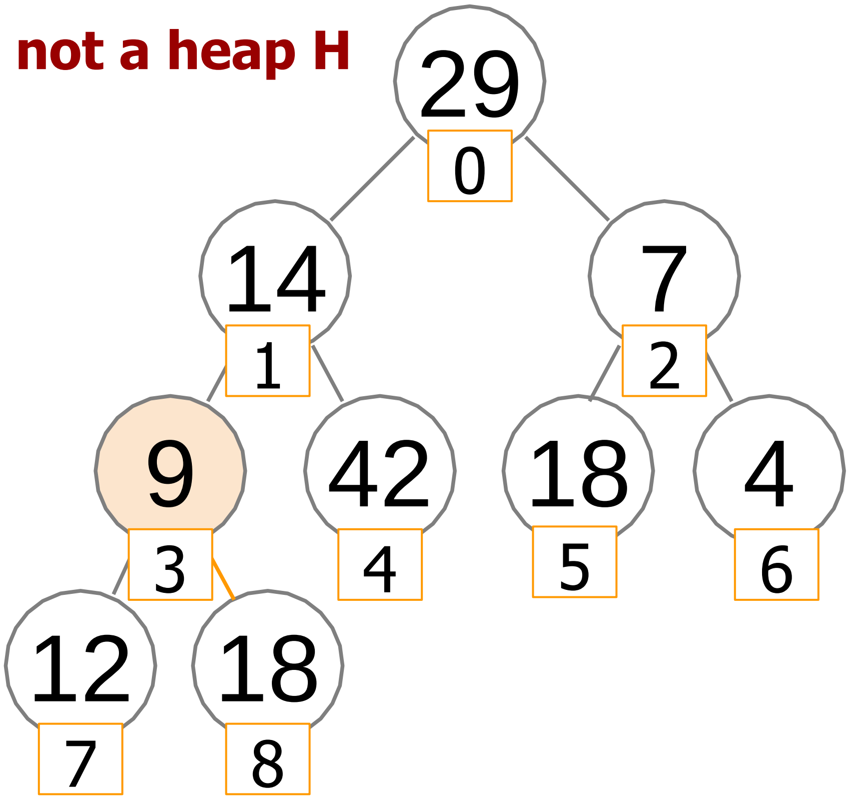


→ We need to process all elements starting from position $i = \lfloor (8-1)/2 \rfloor = 3$ until position 0 and repair heap violations by calling sift_down(i)

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

How to Heapify an array

not a heap H

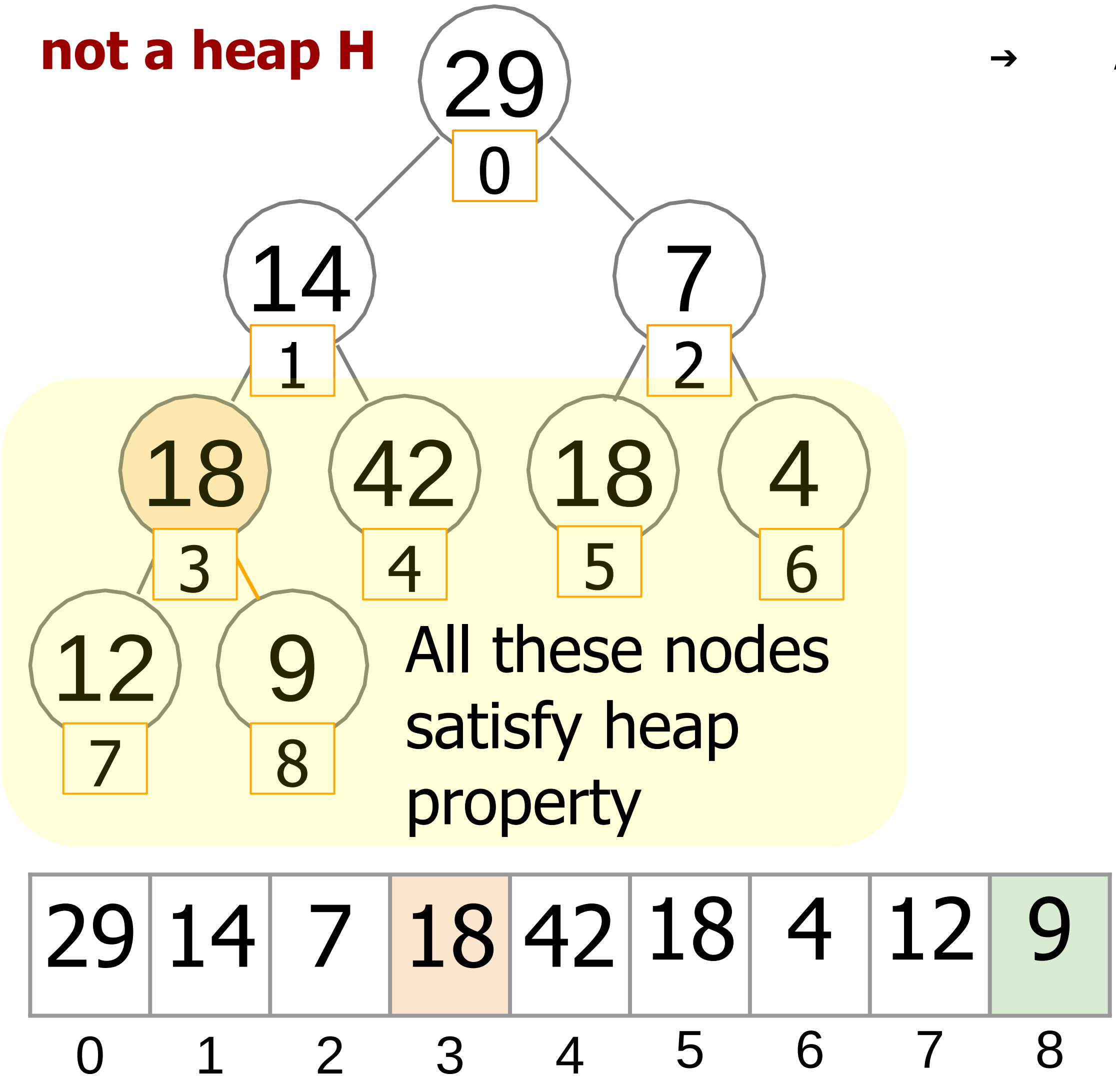


➔ We need to process all elements starting from position $i = \lfloor (8-1)/2 \rfloor = 3$ until position 0 and repair heap violations by calling sift_down(i)

sift_down(3)

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

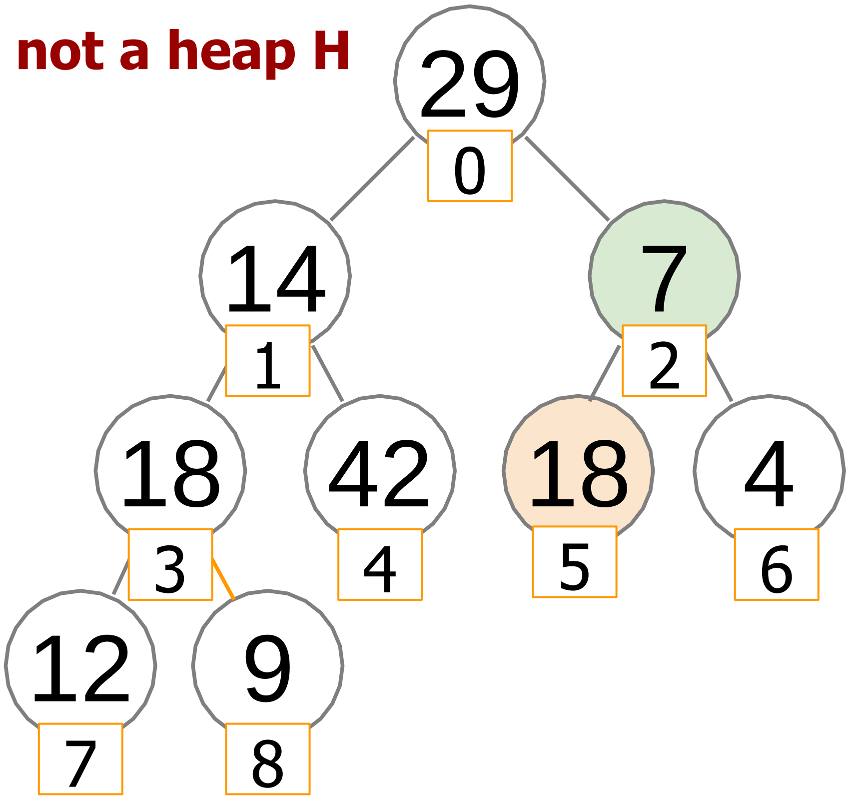
How to Heapify an array



→ All the nodes H[3...8] are now repaired

How to Heapify an array

not a heap H



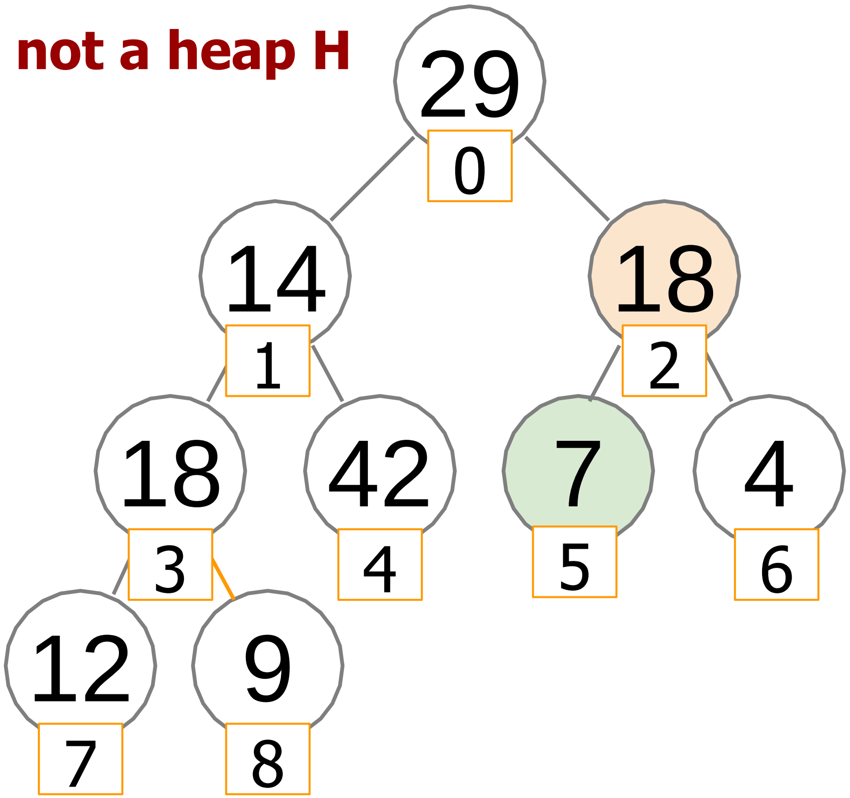
➔ the next node we need to fix is at position 2 of the array

sift_down(2)

29	14	7	18	42	18	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

not a heap H



➔ the next node we need to fix is at position 2 of the array

sift_down(2)

29	14	18	18	42	7	4	12	9
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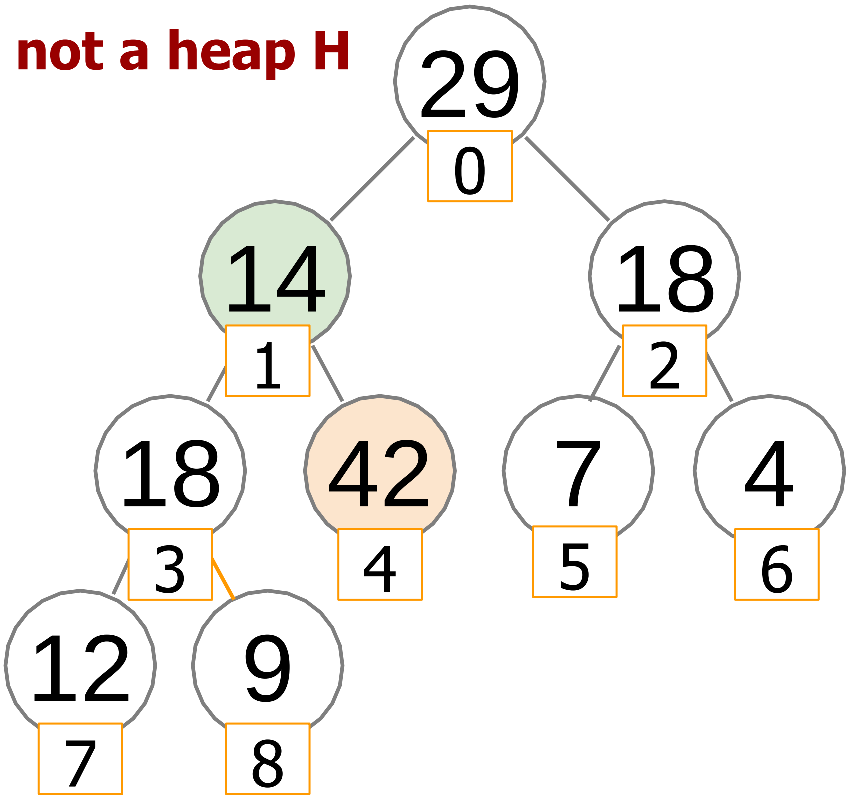
How to Heapify an array

not a heap H



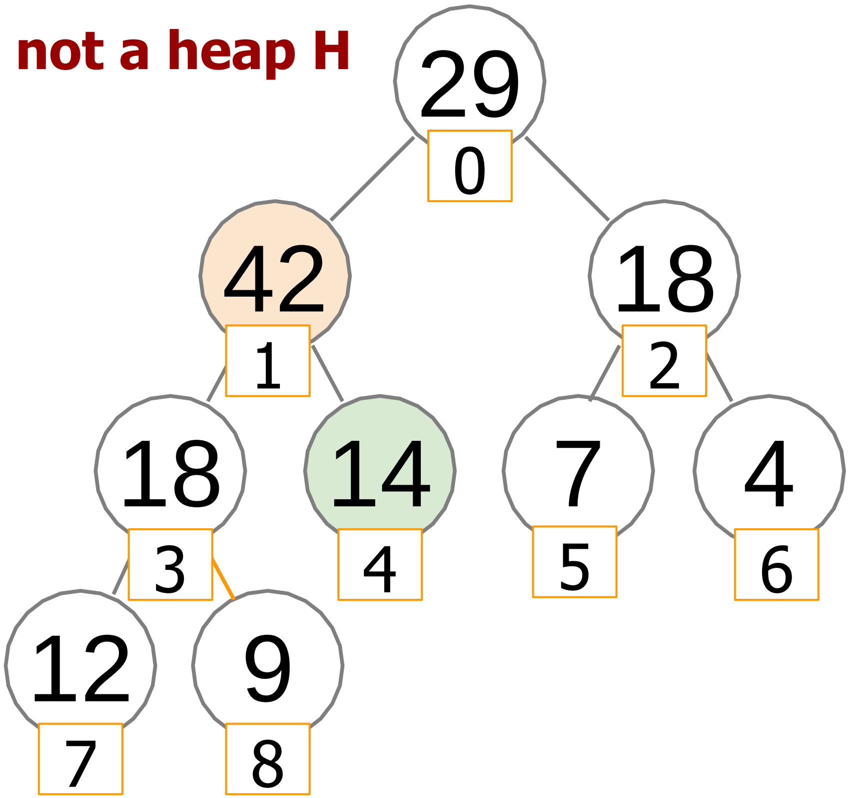
→ the next node we need to fix is at position 1 of the array

sift_down(1)

29	14	18	18	42	7	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

not a heap H



→ the next node we need to fix is at position 1 of the array

sift_down(1)

29	42	18	18	14	7	4	12	9
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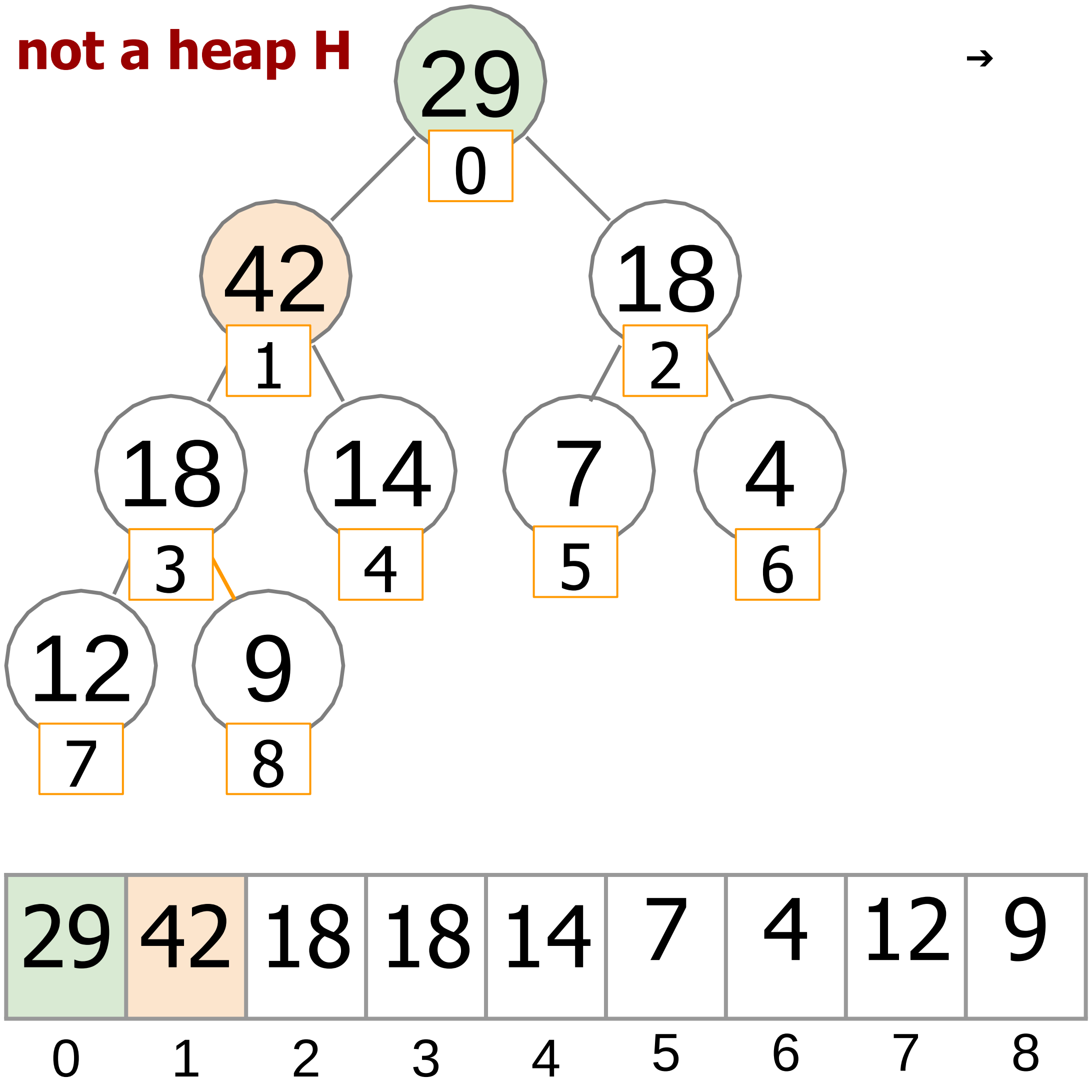
How to Heapify an array

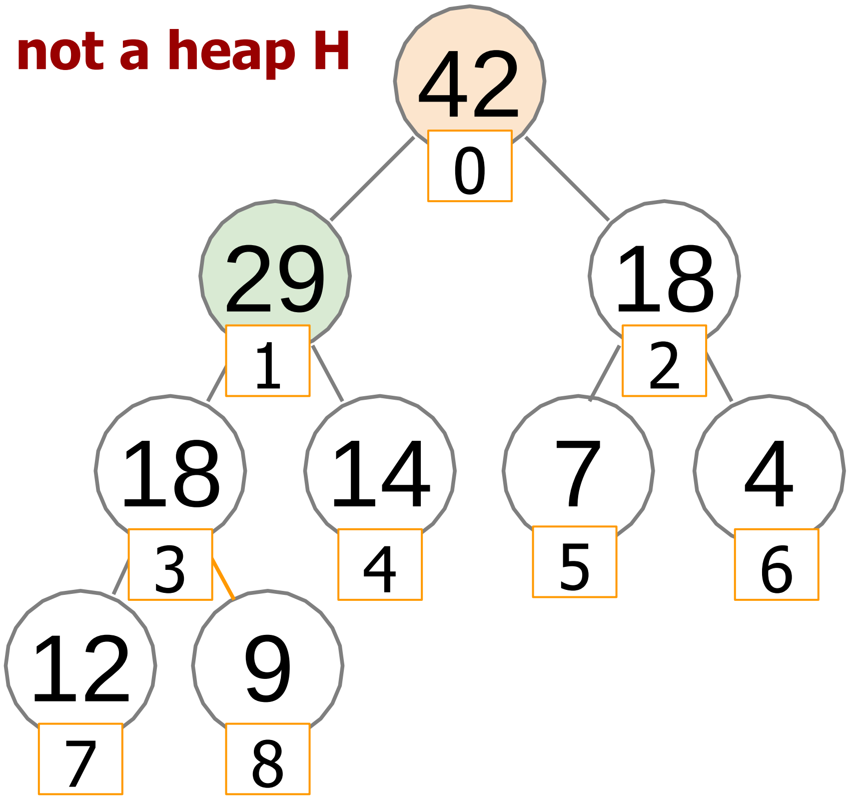


➔ Finally, we fix the root at position 0

sift_down(0)

How to Heapify an array

not a heap H



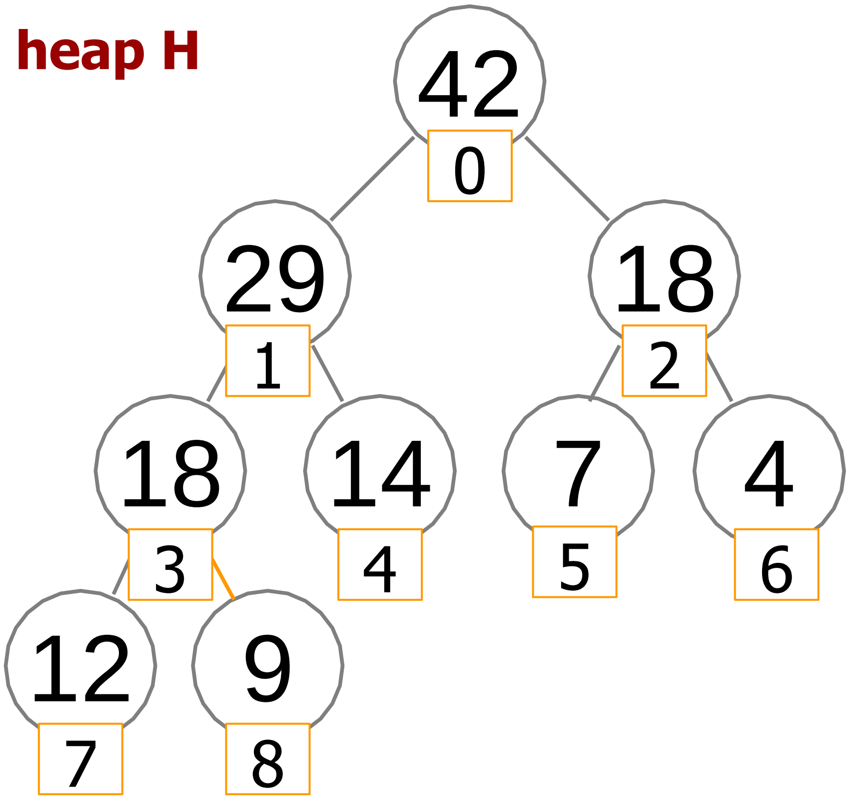
- Finally, we fix the root at position 0

  sift_down(0)

42	29	18	18	14	7	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

heap H



42	29	18	18	14	7	4	12	9
0	1	2	3	4	5	6	7	8

- We rearranged the elements of the input array such that it is now a heap
- Next, we can use *dequeue* inside the array itself to sort it in-place

First - turn Array into a Heap

`Heapify` (array *A* of size *n*)

last ← *n* - 1

for *i* from ⌊(*last -1*)⁄2⌋ down to 0:

    sift_down (*i*)

In-place Heap Sort

HeapSort (array *A* of size *n*)

Heapify (*A*)

m ← n
repeat (n – 1) times:
    swap $A[0]$ and $A[m\text{-}1]$
    $m \leftarrow m - 1$
    sift_down (heap of size m, 0)

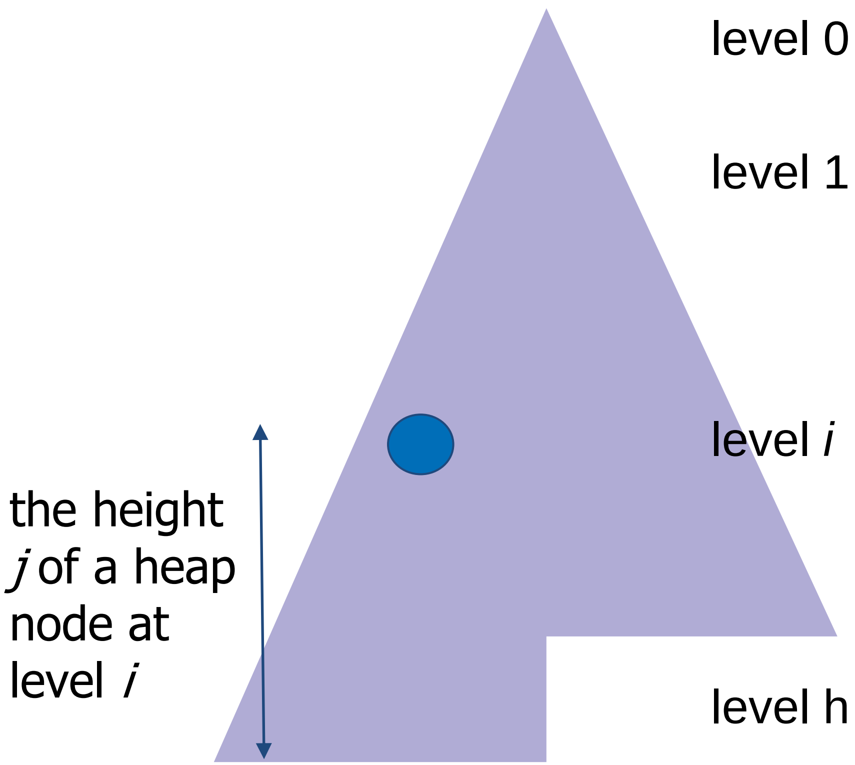
Running time: $O(n \log n)$

No additional space (in-place)

Run-time of Heapify

- The running time of *Heapify* is $O(n \log n)$ since we call *sift_down* for $O(n)$ nodes

The height of nodes at level *i*



Definition

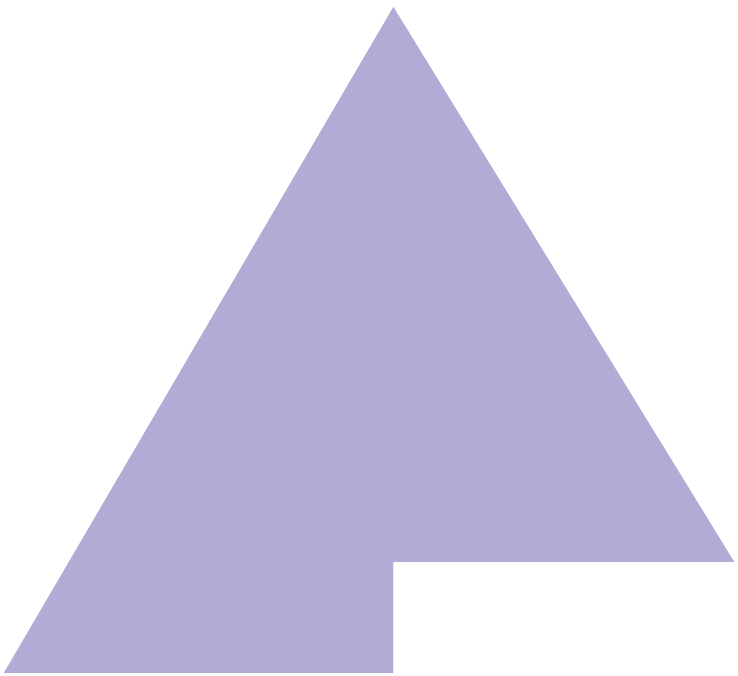
If we count levels of the heap from top to bottom, then the ***height*** of a heap node at level *i* is defined to be $j = h - i$, where *h* is the total height of the heap

When we are repairing the heap, for each node at level *i* we need to swap at most *j* values

Run-time of *Heapify*

- The running time of *Heapify* is $O(n \log n)$ since we call *sift_down* for $O(n)$ nodes
- If a node is a leaf then we do not call *sift_down* on it
- If a node is close to the leaves, then sifting it down does not take $\log n$
- We have many such nodes!
- Is our estimate of the running time of *Heapify* too pessimistic?

Run-time of Heapify



level	# nodes	node height
h-h	2^{h-h}	h
	...	...
h-2	2^{h-2}	2
h-1	$\leq 2^{h-1}$	1
h-0	$\leq 2^{h-0}$	0

Total work:

$$\sum_{j=0}^{h} j * 2^{h-j} = 2^h \sum_{j=0}^{h} j * \frac{1}{2^j}$$

, where *j* represents the height of the nodes at each of 0…*h* tree levels

Commonly used summations

$$\sum_{j=1}^{n} j = \frac{n(n-1)}{2} = O(n^2)$$

$$\sum_{j=0}^{n} 2^j = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1 = O(2^n)$$

$$\sum_{j=0}^{n} d^j = \frac{d^{n+1}-1}{d-1} = O(d^n)$$

$$\sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{1-\frac{1}{2}} = 2 = O(1)$$

$$\sum_{j=1}^{n} \frac{1}{j} \approx \int_1^n \frac{1}{j}\, dj = \ln n = O(\log n)$$

$$\sum_{j=0}^{n} j \frac{1}{2^j} \qquad ?$$

The upper bound of unknown sum

Consider the sum of infinite convergent geometric series with x<1 and a_0=1

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \qquad (1)$$

Take a derivative with respect to x of both parts of equality (1):

$$\sum_{j=0}^{\infty} j * x^{j-1} = \frac{1}{(1-x)^2} \qquad (2)$$

Multiply both sides of (2) by x:

$$\sum_{j=0}^{\infty} j * x^j = \frac{x}{(1-x)^2} \qquad (3)$$

Substitute x=½

$$\sum_{j=0}^{\infty} j * \frac{1}{2^j} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

$$\sum_{j=0}^{\infty} j * \frac{1}{2^j}$$

this sum is at most 2 even for infinite series!

This expression evaluates to O(*n*)

$$2^h \sum_{j=0}^{h} j * \frac{1}{2^j} \leq 2^h * 2 = O(2^h) = O(2^{\log n}) = O(n)$$

The running time of Heapify is ***O(n)***

To convert an arbitrary array into a heap takes linear time and no additional space!

Top-k Problem

Input: An array A of size n, an integer $1 \le k \le n$.

Output: ***k largest*** elements of A (top-k).

Can be solved in time: $O(n) + O(k \log n)$