Kruskal Algorithm: Performance. UNION-FIND

Lecture 05.05

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Kruskal algorithm

Algorithm Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø                              # collects edges of the future MST

for i from 1 to m:
        if T U {E'[i]} has no cycles
                add E'[i] to T

return T
```

Repeatedly add a minimum-cost edge
that does not create a cycle

Kruskal algorithm

Algorithm Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø                          # collects edges of the future MST

for i from 1 to m:
    if T U {E'[i]} has no cycles
        add E'[i] to T

    if |T| = |V| - 1:            # we can stop once we have a tree
        break

return T
```

Stop when
N-1 edges have been selected

Running time

```
Kruskal_MST (graph G(V,E))

1   E' := edges of G sorted by weights
2   T : = Ø

3   for i from 1 to m:
4           if T U {E'[i]} has no cycles
5                   add E'[i] to T

6           if |T| = |V| - 1:
7                   break

8   return T
```

Line 1: sorting m edges by weight. O(m log m). This is the same as O(m log n) Why?

Line 3: outer for loop. O(m). We check all m edges in the worst case.
Line 4: need to find if edge E'[i]= (u,v) creates a cycle.
Find out if there is already a path from u to v in T by any graph traversal (DFS or BFS). DFS of T with n vertices and n-1 edges is O(n + n) = O(n).

Thus, total time of the for loop is O(m)*O(n) = O(mn)

$O(n^3)$ for dense graphs

Kruskal MST runs in time O(m log n) + O(mn) = **O (mn)**

Running time

Kruskal_MST (graph G(V,E))

```
E' := edges of G sorted by weights
T : = Ø

for i from 1 to m:
        if T U {E'[i]} has no cycles
                add E'[i] to T

        if |T| = |V| - 1:
                break

return T
```

Bottleneck: detecting a cycle

Kruskal MST runs in time **O (mn)**

Can we do better?

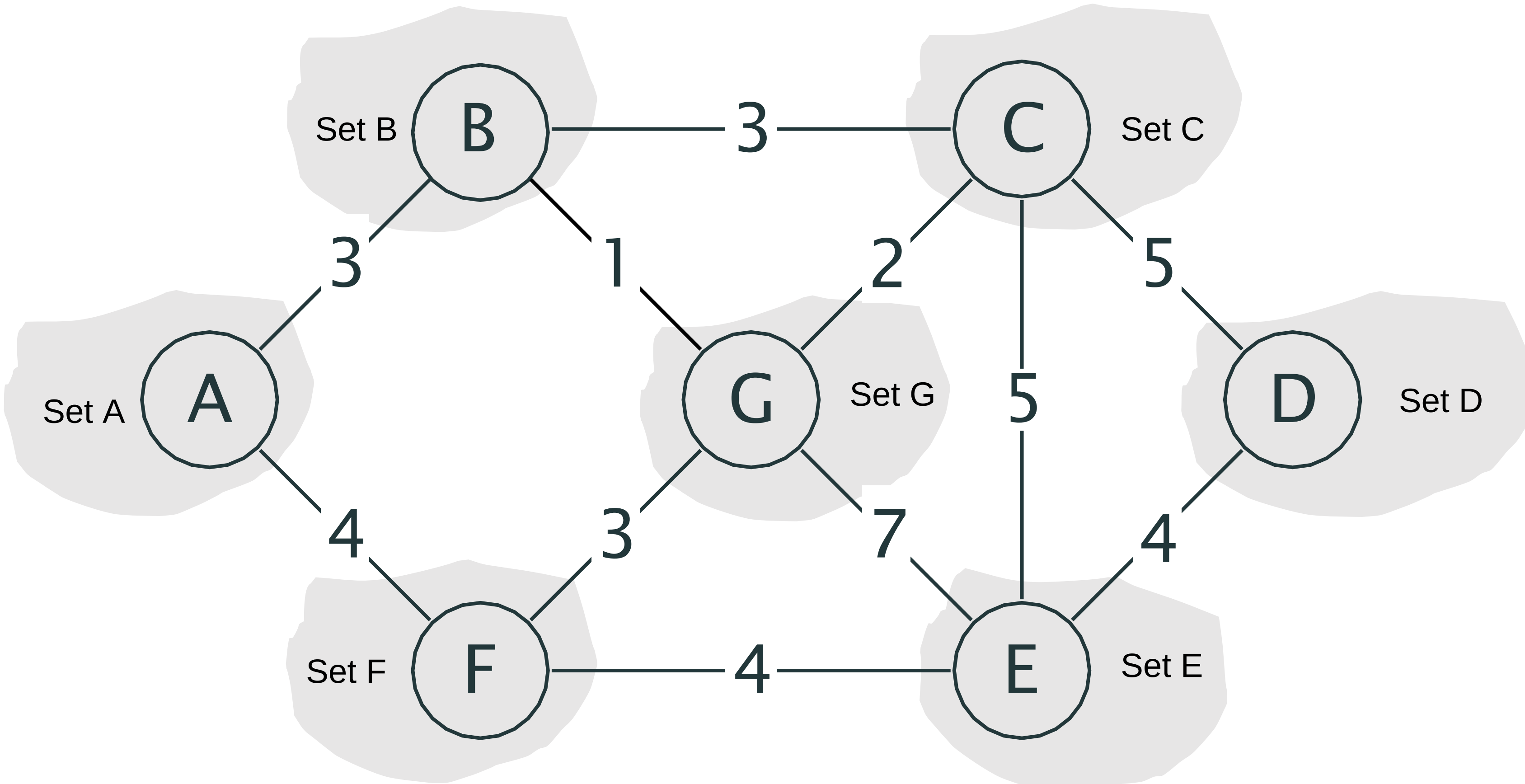
Kruskal as union of sets

We can look at Kruskal from a Set point of view

- First we have n sets: each vertex i is in its own set S_i – we need to be able to MAKE-SET for a single element
- Next we combine two sets of vertices S_i and S_j into one set: we perform UNION (S_i and S_j), adding an edge (u,v) such that $u \in S_i$ and $v \in S_j$
- However we do the union only if $S_i \neq S_j$. In other words, we need to know if u and v are already in the same set, in the same connected component, we need to FIND out set names for u and for v and compare them for equality

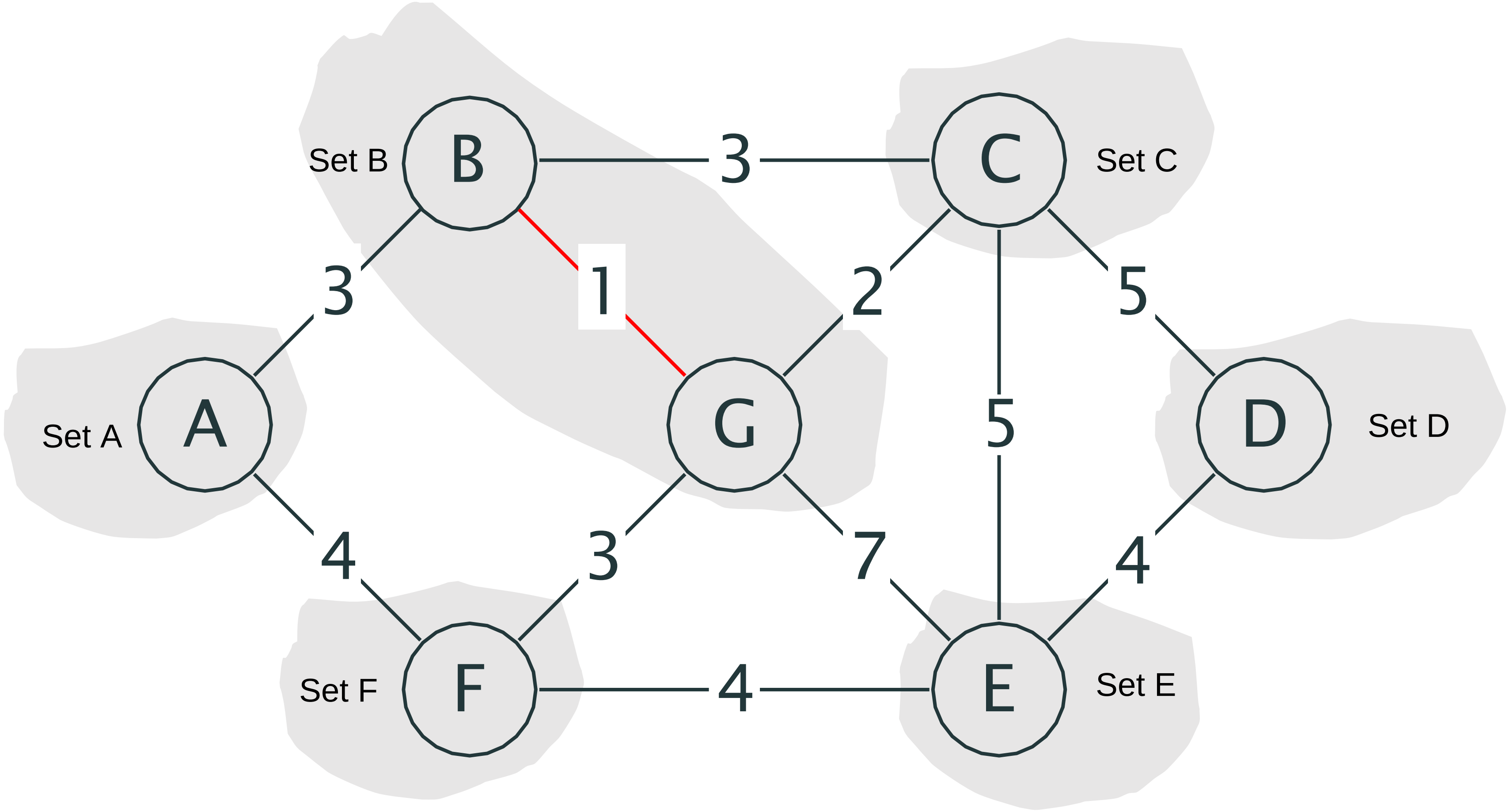
Note that all the sets are *disjoint*: each node belongs to a single set during the execution of the algorithm

Kruskal as union of sets



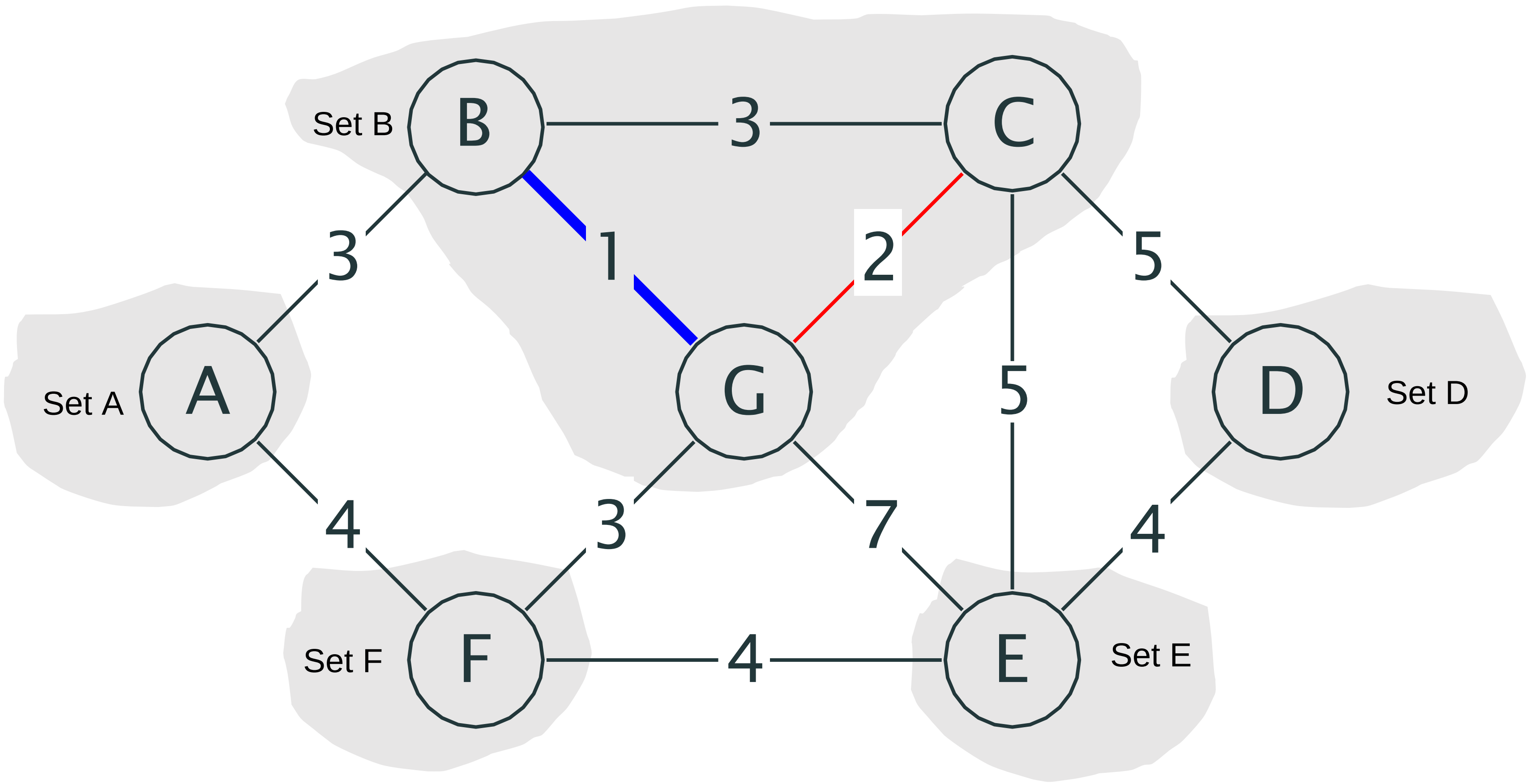
Set B = UNION (B, G)

Kruskal as union of sets



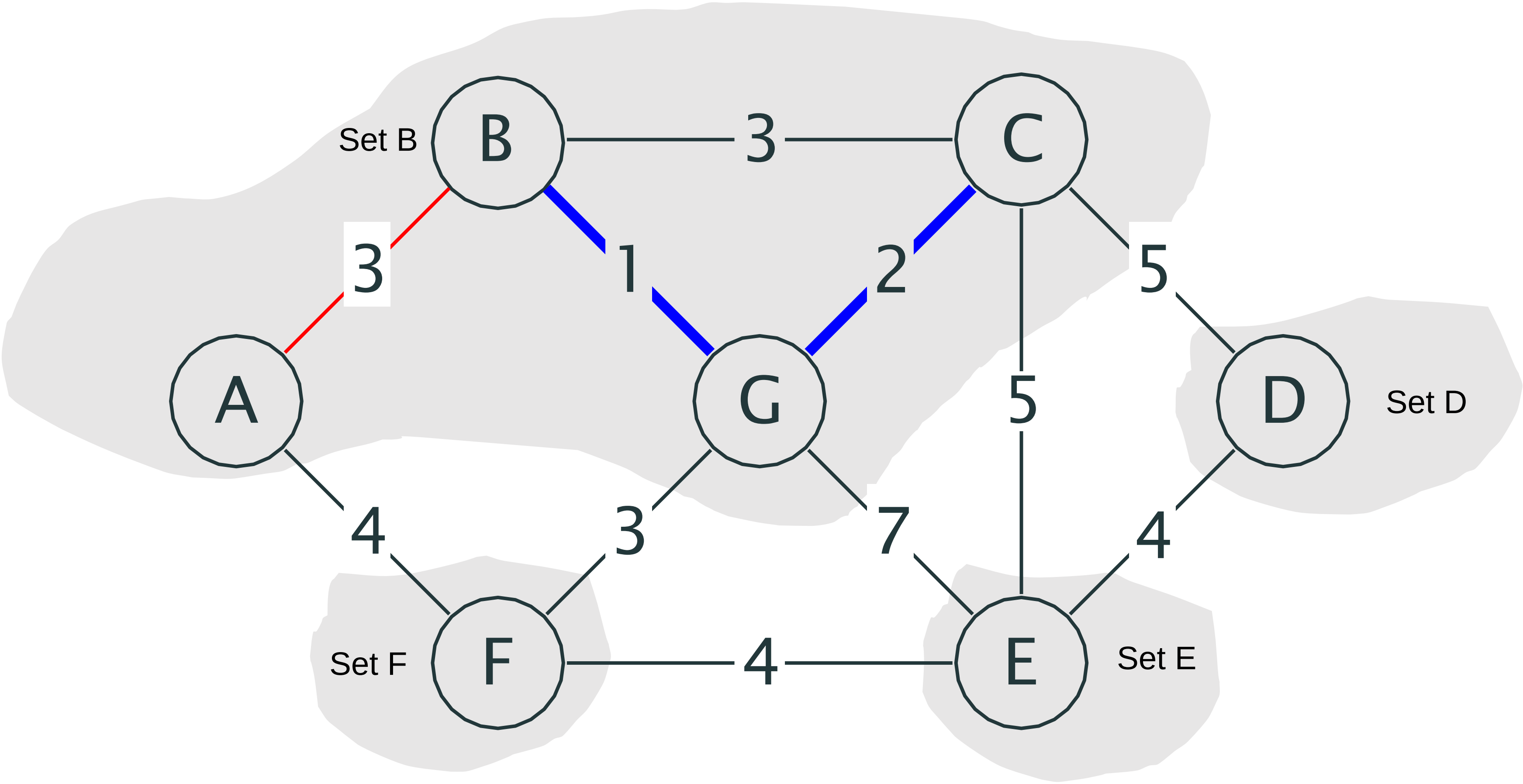
Set B = UNION (B, C)

Kruskal as union of sets



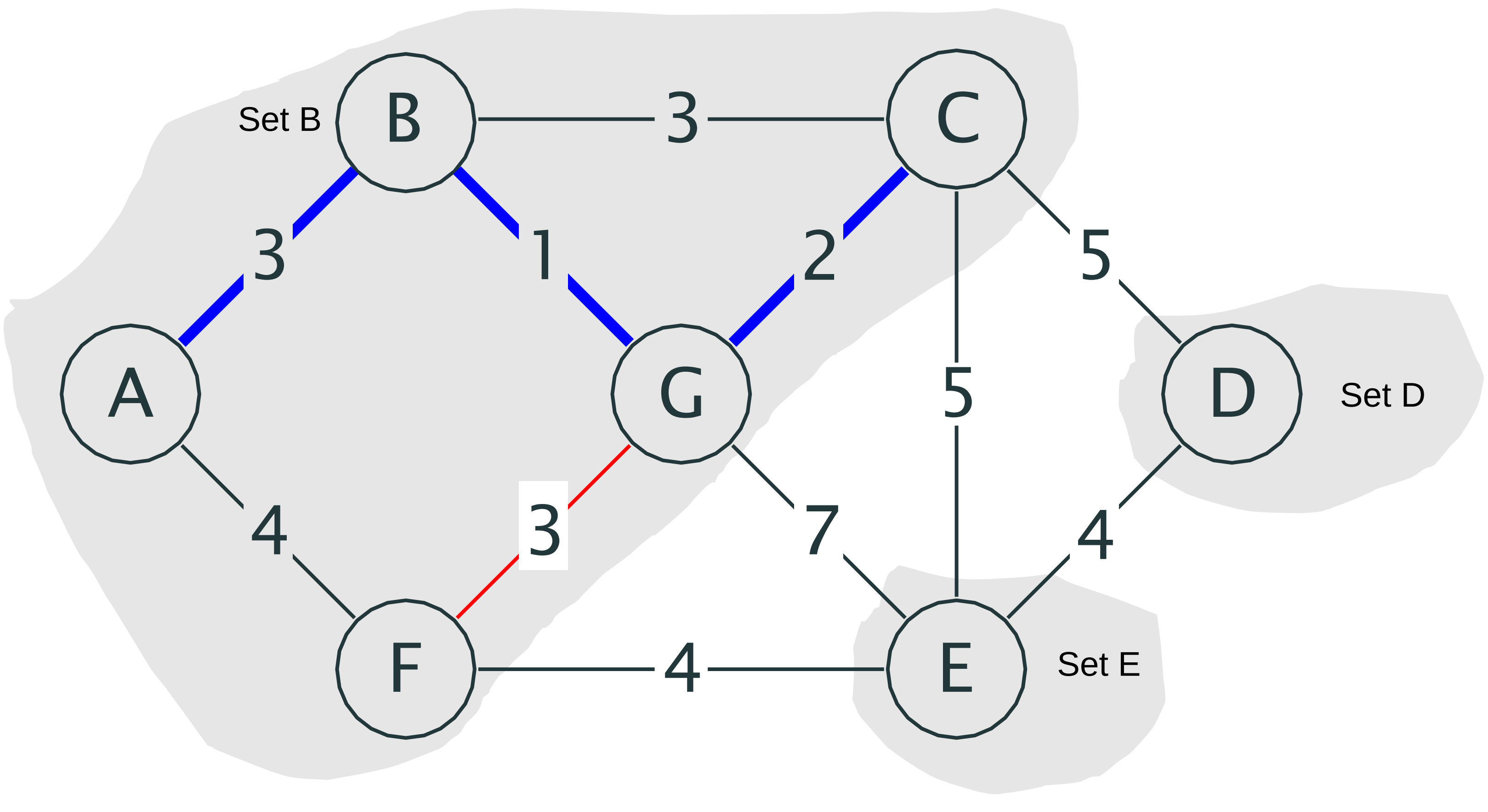
Set B = UNION (B, A)

Kruskal as union of sets



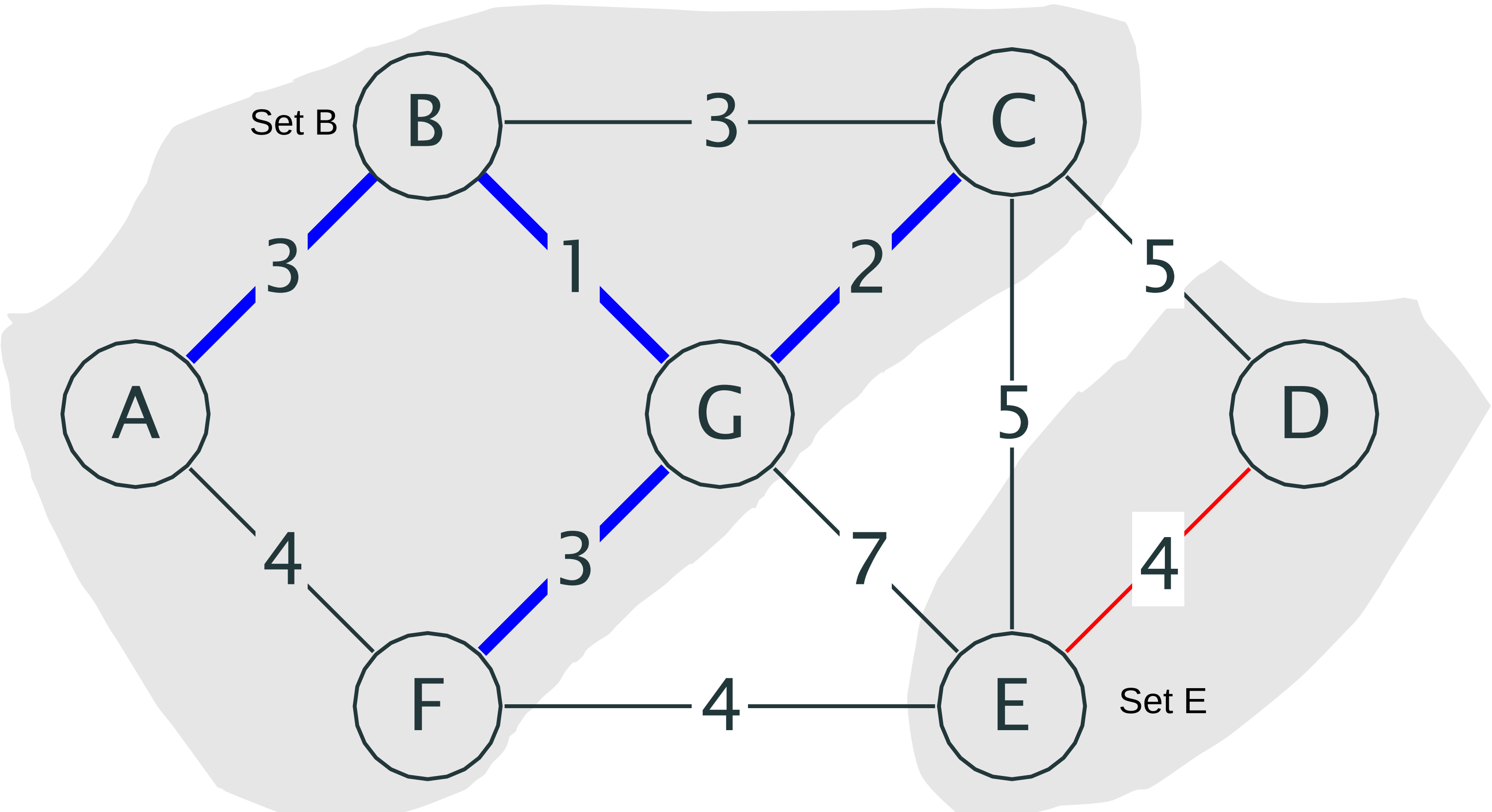
Set B = UNION (B, F)

Kruskal as union of sets



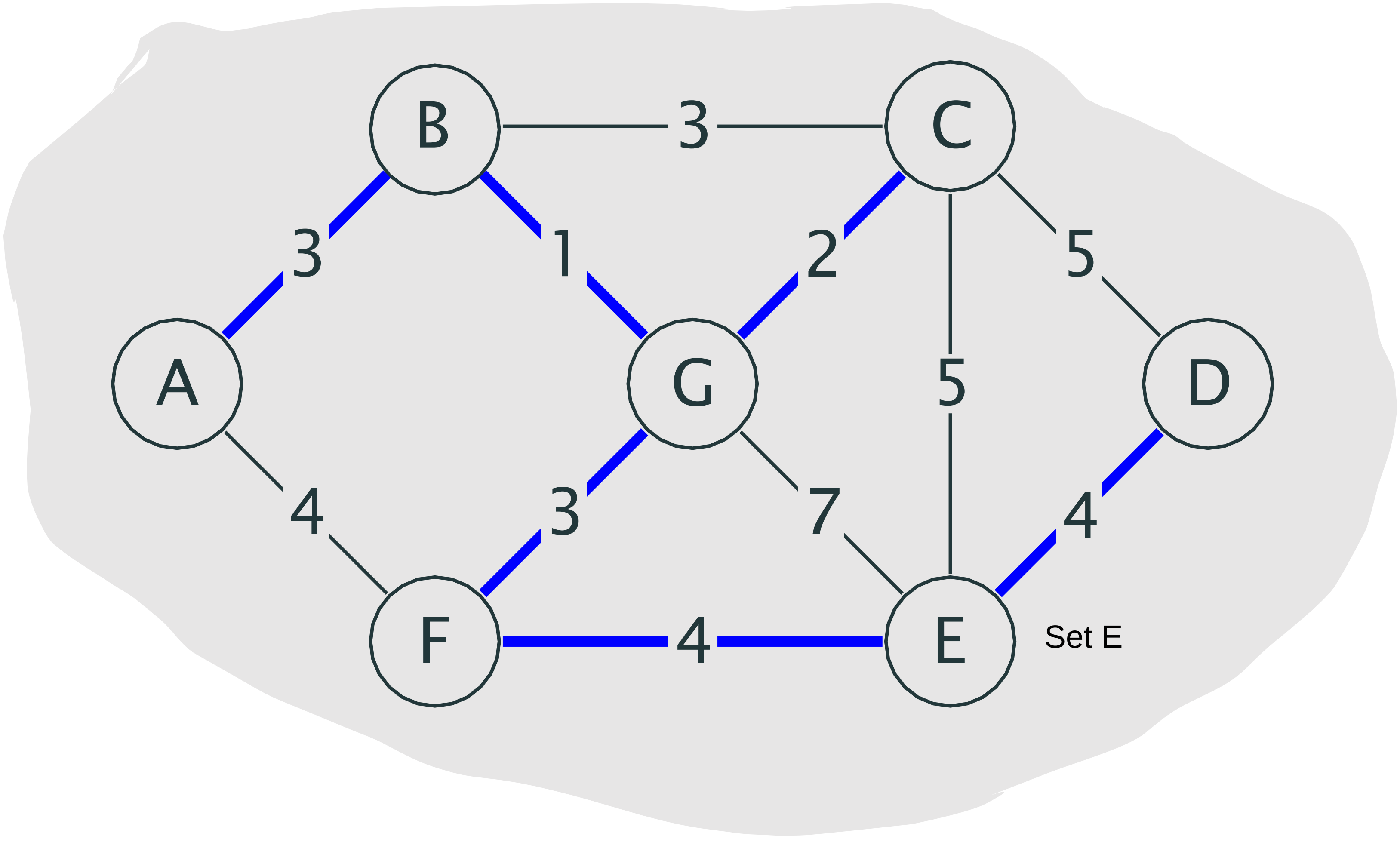
Set E = UNION (D, E)

Kruskal as union of sets



Set E = UNION (B, E)

Kruskal as union of sets



Set spanning all vertices of G with selected edges:
MST of G

New ADT: UNION-FIND (= Disjoint Set ADT)

UNION-FIND is an Abstract Data Type that supports the following operations:

- *MAKESET(x)*: Creates a new set X containing a single element x.
- *UNION(X, Y)*: Creates a new set containing the elements of sets X and Y in their union and deletes the previous sets X and Y.
- *FIND(x)*: Returns the name of the set to which element x belongs.

UNION-FIND fits all our needs

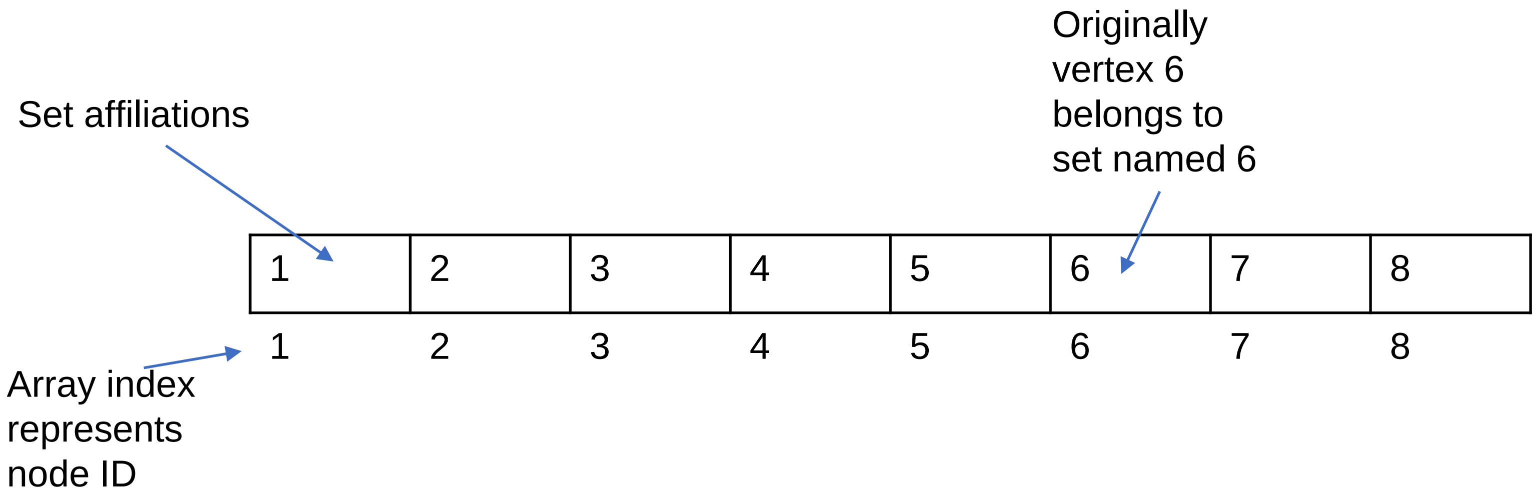
- Initially, the vertices are a collection of n sets, each with one element. We can use *MAKE-SET* n times. Each set has a different element, so that $S_i \cap S_j = \emptyset$. The sets are disjoint.
- To introduce a new edge connecting S_i and S_j using edge (x,y), we first check whether x and y are already connected: perform *FIND*(x) and *FIND*(y) and check if x and y already belong to the same set.
- If they are not, then we apply *UNION*. This operation merges the two sets containing x and y and replaces them with a new set $S_k = S_i \cup S_j$.

Implementing UNION-FIND: Array

- We can implement UNION-FIND using a physical **array**
- We can assign each vertex an ID from 1 to n, and assume that the name of the set to which vertex *i* belongs is stored at position *i* of this array

Array implementation: *MAKE-SET*

- For n elements, we can generate single-element sets in time **O(n)**
- The name of each set initially is set to the name of the element itself: which corresponds to its position i in the array



Index in this array uniquely identifies each of n graph vertices

Array implementation: **super-fast *FIND***

- With this representation FIND(x) takes O(1), since for any element we can find the set name by accessing its array location in time **O(1)**

We can find set affiliations of nodes 3 and 8 in time O(1)

Set affiliation	1	2	6	2	5	6	7	6
Array index	1	2	3	4	5	6	7	8

Array implementation: **slow *UNION***

- To perform UNION(u, v) [assuming that $u \in S_i$ and $v \in S_j$] we need to scan the complete array and change all i's to j. This takes **O(n)** time
- A sequence of n – 1 unions required by the algorithm takes $O(n^2)$ time in the worst case!

Set 1: Contains 1,4,5,7

Set 2: Contains 2,3,6,8

Current set affiliations

1	2	2	1	1	2	1	2
1	2	3	4	5	6	7	8

Next edge to be added: (3,4)
We check that FIND(3) ≠ FIND(4)
UNION(1,2) will need to iterate over the array and replace all 2 with 1

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8

Now vertices 3 and 4 belong to the same set, they are connected

Implementing UNION-FIND: Tree

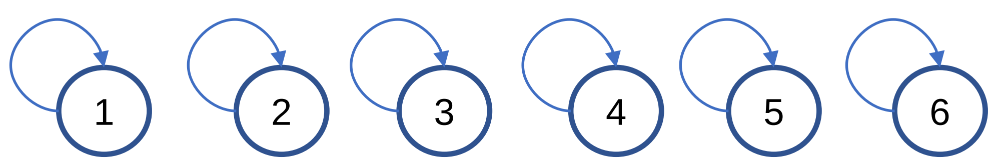
- We can implement each set as a tree, because in the tree each element has only one root, and that is where we will store the name of the set to which all elements in this tree belong

- The tree idea is rather conceptual. We do not have to create a physical tree: we can use a *parent array* where for each node *i* we store the name of its parent in the tree

Tree implementation: *MAKE-SET*

- To differentiate the root of the tree, let us assume that if the parent in position i is i, then node i is a root of the tree – and it also serves as a set name for all nodes in its subtrees
- MAKE-SET creates n sets each containing a single element i and the parent of i is recorded as i. That means root (set name) of i is i.

Parent array	1	2	3	4	5	6
	1	2	3	4	5	6



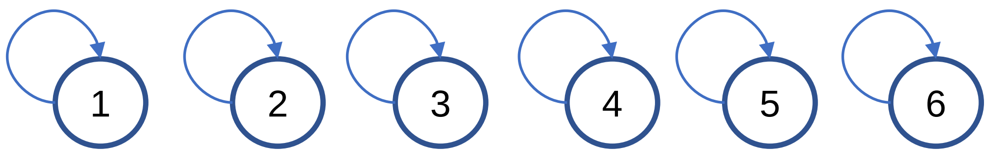
Create a collection of tiny trees, but still store them in the array

Tree implementation: fast ***UNION***

- To replace the two sets containing u and v by their union – update a parent of u to node v

Parent array

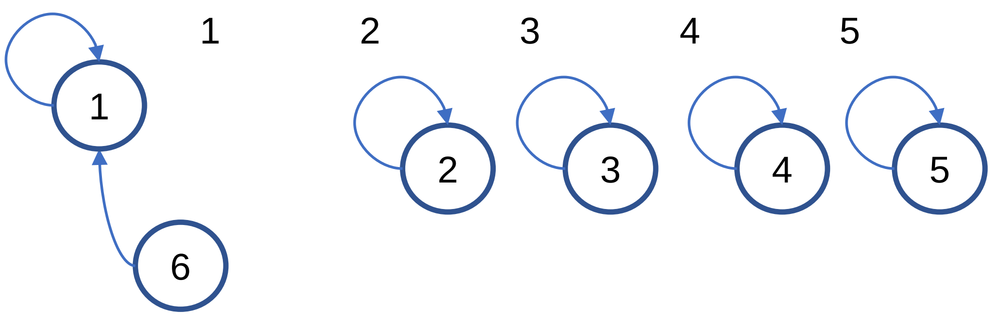
1	2	3	4	5	6
1	2	3	4	5	6



After UNION (1,6)

Parent array

1	2	3	4	5	1
1	2	3	4	5	6

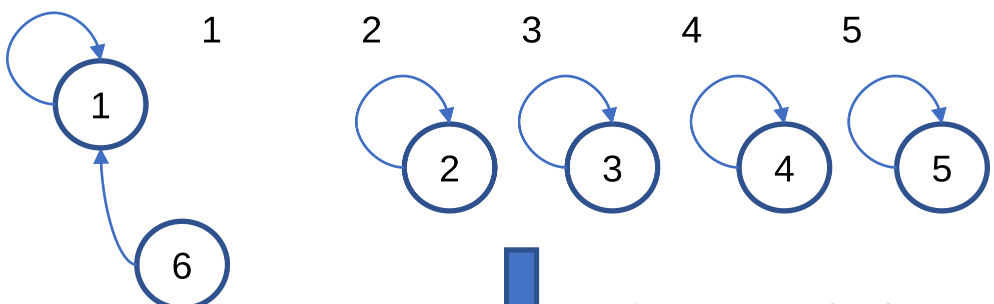


Tree implementation: **fast *UNION***

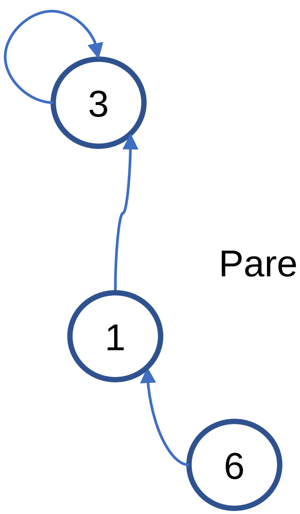
- To replace the two sets containing u and v by their union – update a parent of u to node v

Parent array

1	2	3	4	5	6
1	2	3	4	5	1



After UNION (3,1)



Parent array

1	2	3	4	5	6
3	2	3	4	5	1

Tree implementation: **fast *UNION***

- To replace the two sets containing u and v by their union – update a parent of v to node u
- Important to note: UNION operation is changing the root's parent only, but not the parent for all the elements in the second set
- Therefore, the time complexity of UNION is **O(1)**

Tree implementation: **slow *FIND***

- A FIND(x) on node x is performed by returning the root of the tree containing x
- The time to perform this operation is proportional to the depth of the node representing x
- It is possible to create a tree of depth n - 1 (Skewed Tree).

- Hence, the worst-case running time of a FIND is **O(*n*)** and m consecutive FIND operations take O(*mn*) time in the worst case. (not an improvement comparing to O(n) DFS algorithm to check for a cycle that we had before)

The goal:
Fast *UNION* + **Quick *FIND***

- The main problem with the previous approach is that we might get skewed trees and as a result the FIND operation takes O(n) time
- We want to keep the height of each tree at most log n

UNION by Size

Simple heuristic:

- Always make the smaller tree a subtree of the larger tree

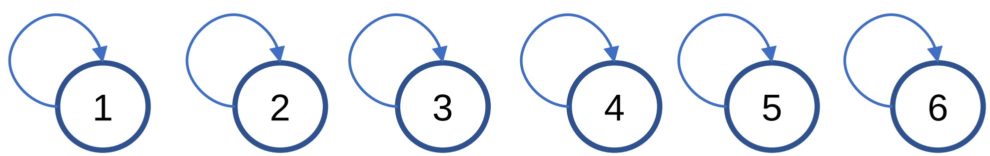
We use the same parent array

- We identify the **root** element of each tree by storing a **negative integer** representing the size of the tree rooted at node i

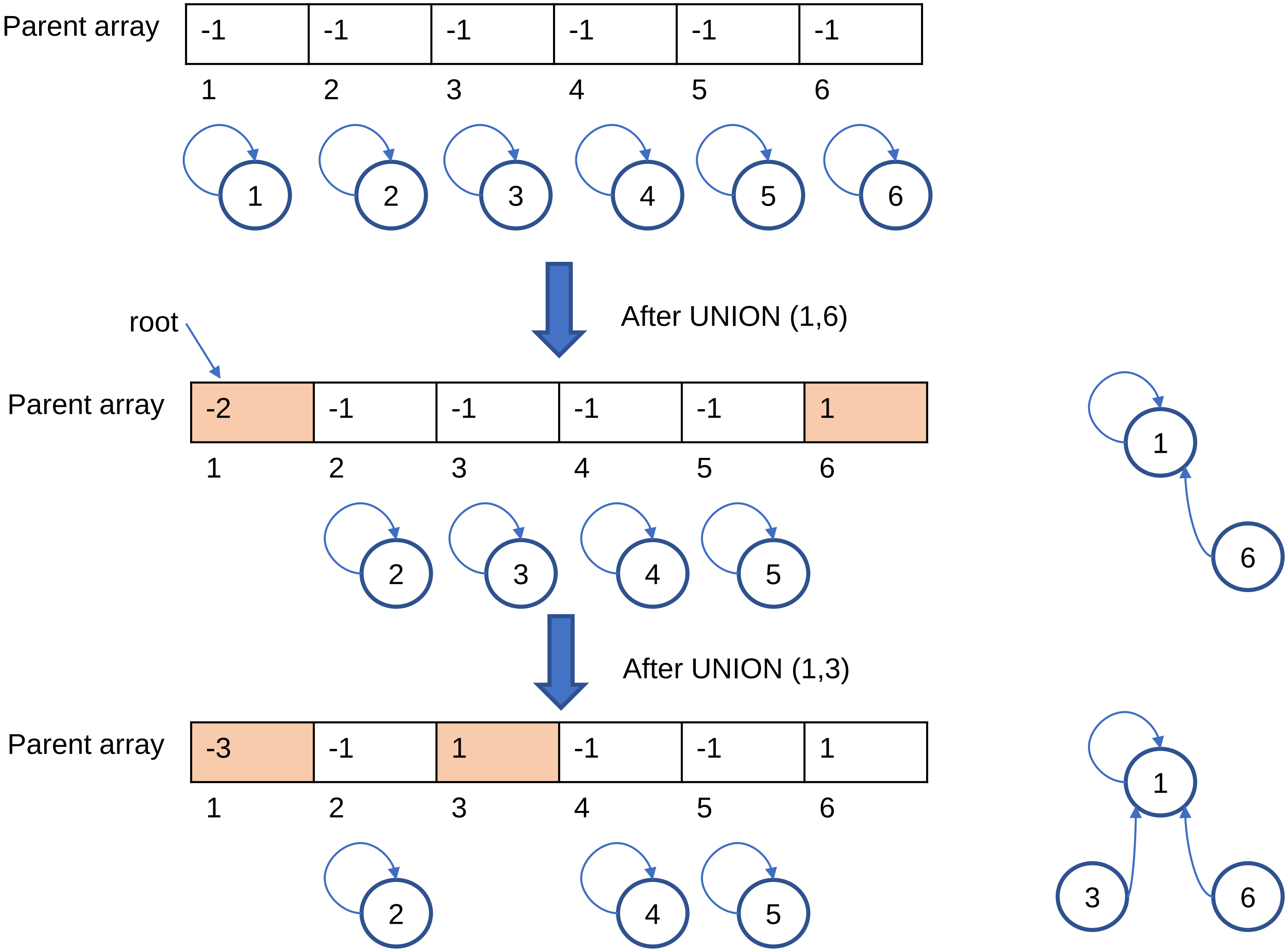
After n calls to MAKE-SET

Parent array

-1	-1	-1	-1	-1	-1
1	2	3	4	5	6

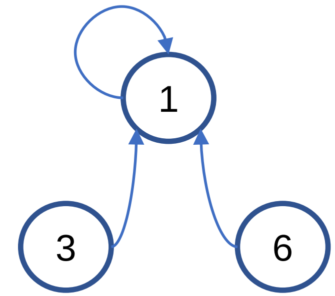


Each node is a parent of a tree. Each tree has size 1


Parent array
-1 -1 -1 -1 -1 -1
1 2 3 4 5 6
1 2 3 4 5 6
After UNION (1,6)
root
Parent array
-2 -1 -1 -1 -1 1
1 2 3 4 5 6
2 3 4 5
1
6
After UNION (1,3)
Parent array
-3 -1 1 -1 -1 1
1 2 3 4 5 6
2 4 5
1
3 6

Parent array

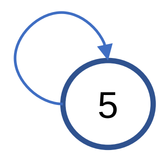
-3	-1	1	-1	-1	1
1	2	3	4	5	6

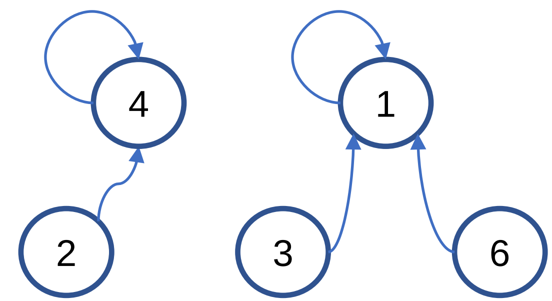


After UNION (4,2)

Parent array

-3	4	1	-2	-1	1
1	2	3	4	5	6

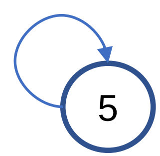


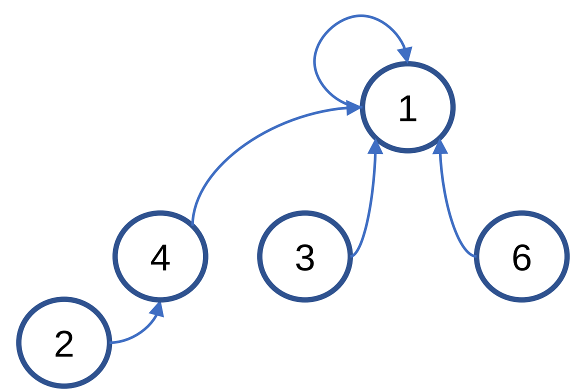


After UNION (4,1)

Parent array

-5	4	1	1	-1	1
1	2	3	4	5	6





UNION by size: **quick FIND**

- With UNION by size, the depth of any node is never more than log n. This is because each node is initially at depth 0. When its depth increases as a result of a UNION, it is placed in a tree that is at least twice as large as before.
- That means that the depth of each node can be increased at most log n times until it becomes a part of a tree with n nodes (there are at most log n UNIONs per each node).
- This gives the running time for a FIND operation as **O(log n)**
- A sequence of m FINDs and UNIONs takes O(m log n).

There are other methods that achieve the same and even better performance

- UNION by Height (UNION by Rank)
- Path Compression
- …

You do not have to know all of them for this course

Running time of UNION-FIND ADT implemented as a Tree (parent array)

Operation	
MAKE-SET(x)	**O(1)**
FIND(x)	**O(log n)**
UNION(x,y)	**O(1)**

Fast UNION – **Quick FIND**

Kruskal running time with UNION-FIND

Kruskal_MST (graph G(V,E))

```
1   E' := edges of G sorted by weights
2   T : = Ø

3   for  i from 1 to n:
4           MAKE-SET (node i)

5   for each edge (u,v) in E':
6           if FIND(u) ≠ FIND(v):
7                   T: = T U (u,v)
8                   UNION(u, v)

9           if |T| = |V| - 1:
                    break

    return T
```

Line 1: sorting m edges by weight. O(m log n).

Line 3: Making an array of size n: O(n).

Line 5: O(m) edges in the worst case.
For each edge: perform FIND O(log n) and

sometimes UNION in time O(1)

Thus, total time of the for loop is O(m log n)

Kruskal MST with UNION-FIND runs in time O(m log n) + O(n) + O(m log n) = **O (m log n)**